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AN EXTENSION OF FERMATEAN B-DERIVED FUZZY SOFT TERNARY SUBGROUP

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ABSTRACT. In this paper, an extension of the fermatean uncertainty soft subgroup structures under a norm. Also, the cubic fermatean uncertainty soft ideal structures and fermatean uncertainty multigroup over multi-homomorphism are discussed in detail.

1. INTRODUCTION

The idea of fuzzy uncertainty as a major role occurs in all fields of science and engineering domains and it was initially developed by Zadeh [17]. Many scientists concentrated the fuzzy uncertainty to predict the fuzzified solution of the applications. The idea of fuzzification in group into uncertainty subgroup is introduced by Rosenfeld [14]. This is the first uncertainty filation of any quantity of algorithmic structures and newly expressed in another dimension in the different fields of science and engineering. The uncertainty is initially discussed through the n-ary systems by Kasner [7] and Dudek [2]. Furthermore, extension of this uncertainty on n-ary groups introduced by Dornte [6]. The uncertainty of n-ary groups as a generalization of uncertainty subgroup discussed by Davvz in [5]. Atanassov [1] introduced the axioms of set properties with notations in uncertainty sets. Dudek [3] has established the axioms of set properties along with basic notations using n-ary systems. Investigate of uncertainty n-ary subgroups introduced by [2]. The fermatean uncertainty sets was initially introduced by Senapathi and Yager [15]. D. Molodtsov's introduced the concept of soft sets in [11]. Followed by, the cubic fermatean uncertainty soft ideal structures, defining basic notions in soft sets along with suitable applications were presented in [8]. R.Nagarajan studied fermatean uncertainty multi-group over multi-homomorphism [12]. In this paper, an extension of the fermatean uncertainty soft subgroup structures under a norm. Also, the cubic fermatean uncertainty soft ideal structures and fermatean uncertainty multi-group over multi-homomorphism are discussed in detail. The rest of the article is structured as follows: Section 2 explained the preliminaries related to the present study. Section 3 presented an extension fermatean uncertainty under normal

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subgroups such important theoretical proofs, corollaries and numerical examples related to these results. Conclusions and future research directions are presented in Section 4.

2. PRELIMINARIES

Let G be a non-empty set. Define (G, ρ) be an n-dimensional groupoid as a mapping $\rho : G_n \to G$, $n \geq 2$. Let $y_i, y_{i+1}, y_{i+2}, y_{i+3} \dots y_j$ be the sequence of elements in n-dimensional groupoids denoted by y_j^i is a set of symbols $y_i = y_{i+1} = y_{i+2} = y_{i+3} \dots =$ y commonly denoted by y^t. Mathematically, we write $\rho(y_1, y_2, \dots, y_n) = \rho(y_1^n)$ and $\rho(y_1, y_2, \dots, y_i, \dots, y_n, \dots, y_{(i+t+1)}, \dots, y_n) = \rho(y_1^t, \dots, y_2^t, \dots, y_{(i+t+1)}^n)$.
For any $y_1, y_2, \dots, y_{n-2} \in G$ then n-dimensional groupied (G, ρ) is called (i, j) associative if it satisfies $\rho(y_i^{i-1}), \rho_i^{(n+i-1)}, y_{(n+1)}^{(2n-1)} = \rho(y^{(j-1)_i}, \rho(y_j^{(n+j-1)}, y_{(n+j)^{(2n-1)}})$. For all the above operation is associative then (G, ρ) is said to be an n-dimensional semi group. Moreover, the n-dimensional groupoid satisfies associative condition if and only if it is called (i, j) - associative for all$j = 2, 3, \dots, n$. In binary case, the value of n=2 is called usual semi group. $y_0, y_1, y_2, y_3 \dots y_n \in G$ with an integer $i \in \{1, 2, 3, \dots n\}$, there exists an object $z \in G$ such that

$$\rho(y_i^{(i-1)}, z, \dots, y_{(i+1)}^n) = y_0 \tag{2.1}$$

The above Eqn. (1) is known as i-solvable or solvable at the place 'i'. If the solution is unique then Eqn. (2.1) is uniquely i-solvable.
For, $i = 1, 2, 3, \dots, n$ an n-dimensional groupoid (G, ρ) is uniquely solvable then it is called an n-dimensional quasigroup. An n-dimensional quasigroup satisfies the associative property then it is called n-dimensional group. For $n \geq 3$ the element in an n-dimensional operation ρ defines binary operation $y \bullet x = \rho(y, c_2^{n-2})$. If$(G, \rho)$ is an n-dimensional group the $(G, \bullet)$ is a group. Followed by the nature of different groups were obtained based on the element c_2^{n-2} . In all cases, these groups are isomorphic to each other [4]. So, we can consider only the groups of the form $ret_c(G, \rho) = (G, \bullet)$ satisfies the operation $y \bullet x = \rho(y, c_2^{n-2}, x)$. In this group, take $e = c$. Then $y^t = \rho(\bar{c}, c^{n-3}, y, \bar{c})$. Dudek and Michalski et. al [4] proved the result is as follows: For the elements $a, y_1^n \in G$ in an n-dimensional group(G, ρ) there exists a group $(G, \bullet)$ with automorphism satisfies

$$\rho(y_i^n) = y_1 \bullet \phi(x_2) \bullet \phi^2(x_2) \bullet \phi^3(x_3) \bullet \phi^{(n-1)}(x_n) \bullet a \tag{2.2}$$

Senapati et. al [15] Let X be a universe of discourse. The operation F in X is a fermatean uncertainty set (FUS) defined by $F = \{\langle x, m_F(x), n_F(x)\rangle / x \in X\}$ with corresponding membership functions $m_F(x) : X \to [0, 1]$ and $n_F(x) : X \to [-1, 0]$ satisfies $0 \leq (m_F(x))^3 + (n_F(x))^3 \leq 1 \ \forall x \in X$. Here in, $m_F(x), m_F(x)$ denotes the degree of membership and non-membership of an element $x \in F$.

Senapati et. al [15] For any FUS, the elements x, $F \in X$ the degree of an indeterminacy is given by $\Pi F(x) = \sqrt[3]{1 - (m_F(x))^3 - (n_F(x))^3}$. The ordered pair of the membership functions $(m_F(x), m_F(x))$ is the fermatean uncertainty number (FUN) denoted by (m_F, n_F).

Senapati and Yager et. al [15] The degree of the set of fuzzy membership functions are more than the degrees of both set of Pythagorean membership grades and bi-uncertainty membership grades. These membership grades are graphically plotted in Figure 2. In Figure 1, the set of points on the line $x + y \leq 1$ shows the bi-uncertainty membership values. Moreover, the set of points that covers from the line $x + y \leq 1$above to atmost $x^2 + y^2 \leq 1$ shows the degree of Pythagorean membership values.

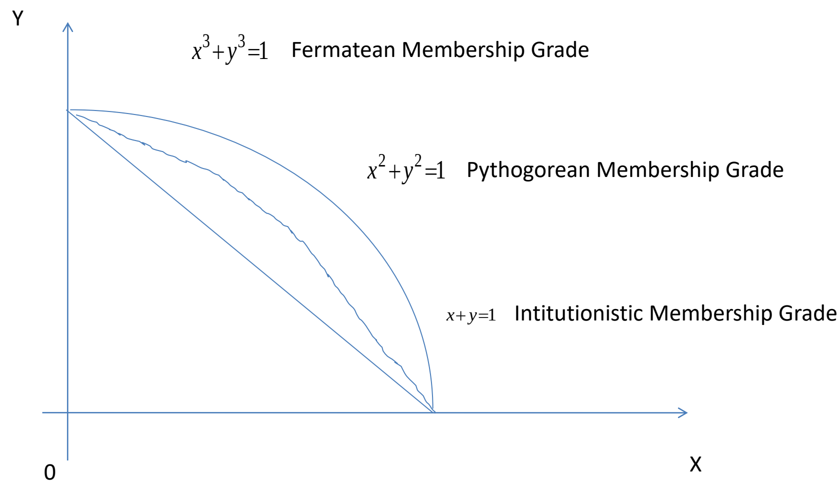

FIGURE 1. Fermatean membership grade

Davvaz et. al [5] An extension uncertainty subgroup $(EUSG)$ in an n-dimensional group on (G, ρ) satisfies following axioms
$[EUSG_1]$: For all $y_1^n \in G, \mu(\rho y_1^n) \geq min\{\mu y_1, \mu y_2, \ldots, \mu y_1\}$
$[EUSG_2]$: For all $y_1^n \in G, \mu(y_1^0) \geq \mu(y)$

For every $y \in G$, there exists a natural number P such that $y^{-(p)} = y$ where $y^{-(p)}$ indicates the skew elements$y^{-(p-1)}$ and $y^{-(0)}$ Dudek et.al: describes an uncertainty over n-dimensional subgroups satisfies $\mu(\overline{y}) > \mu(y)\ \forall y \in G$. Followed by the above backgrounds scope of the study, we discussed the theorems in Fermatean uncertainty of normal subgroups with suitable numerical examples.

3. EXTENSION FERMATEAN UNCERTAINTY OF NORMAL SUBGROUPS

In this section, an extension fermatean uncertainty (T, S) normal subgroup with related theorems, corollary with suitable examples are discussed in detail.

Definition 3.1. A collection of Fermatean uncertainty $A = (m_A^3, n_A^3)$ in G is said to be an extension fermatean uncertainty (T, S) under normal subgroup of satisfies the following axioms:

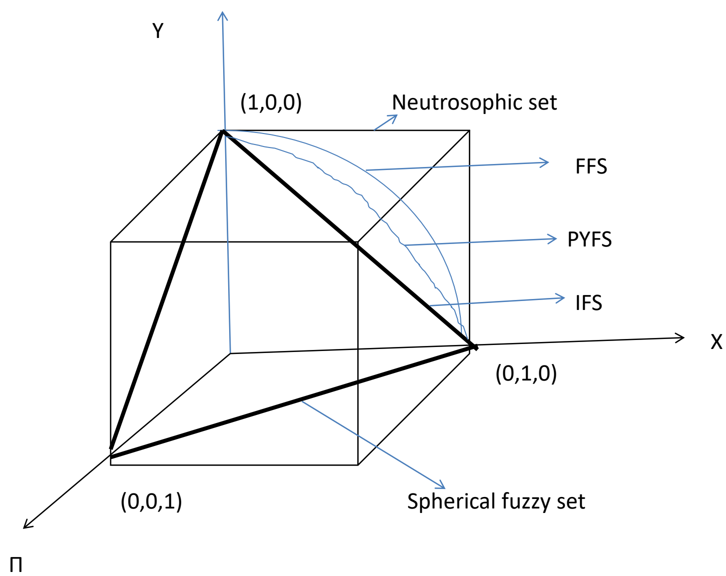

FIGURE 2. Types of membership functions

$$EFUSG_1 : For\ all\ y_1^n \in G \Rightarrow (m_A^3(\rho(y_1^n)) \geq T\{m_A^3(y_1), ..., m_A^3(y_n)\})$$
$$EFUSG_2 : For\ all\ y_1^n \in G \Rightarrow (n_A^3(\rho(y_1^n)) \leq S\{n_A^3(y_1), ..., n_A^3(y_n)\})$$
$$EFUSG_3 : For\ all\ y_1 \in G \Rightarrow (m_A^3(y) \geq m_A^3(y))$$
$$EFUSG_3 : For\ all\ y_1 \in G \Rightarrow (m_A^3(y) \leq m_A^3(y))$$
$$EUSG_2 : For\ all\ y_1^n \in G \Rightarrow \mu(\bar{y}) \geq \mu(y)$$

Example 3.2. Let (Z_4, ρ) be an extension Fermatean uncertainty subgroup. Define a mapping $\rho : Z_4^4 \to Z_4$ defined by $\rho(y_1, y_2, y_3, y_4) = S(y_1, y_2, y_3, y_4)$.
Consider, (Z_4, ρ) is a 4-dimensional additive subgroup of an integer modulo 4.
Let $A = (m_A^3, n_A^3)$ be the fermatean uncertainty collection in (Z_4, ρ) given by

$$m_A^3 = \begin{cases} 0.8, & if\ y = 0 \\ 0.3, & if\ y = 1, 2, 3 \end{cases} \quad \text{and} \quad n_A^3 = \begin{cases} 0.3, & if\ y = 0 \\ 0.8, & if\ y = 1, 2, 3. \end{cases}$$

The above fermatean uncertainty collection $A = (m_A^3, n_A^3)$ satisfies an extension fermatean uncertainty subgroup of (Z_4, ρ).

Theorem 3.1. *If $\{A_i / i \in I\}$ is an arbitrary family of an extension fermatean uncertainty subgroup of (G, ρ) then $\bigcap A_i$ is also an extension fermatean uncertainty subgroup of (G, ρ) given by*

$$\bigcap A_i = \{y, \big(\wedge m_A^3(y), \vee n_A^3(y)\big) / y \in G\}.$$

Proof. According to Definition 3.1: the proof of this theorem obviously satisfied. □

Theorem 3.2. *If $A = (m_A^3, n_A^3)$ be a collection of fermatean uncertainty in G is an extension fermatean uncertainty subgroup of (G, ρ) with operation $\diamondsuit$ satisfies $A = \{y, (m_A^3(y)), 1 - n_A^3(y))/y \in G\}$.*

Proof. From Definition 3.1 : the membership function $\bar{m}_A$ satisfies the axioms $EFUSG_2$ and $EFUSG_4$.

Let $y_1^n \in G$ then the membership function can be written as

$$\begin{aligned}\bar{m}_A^3(\rho(y_1^n)) &= 1 - m_A^3(\rho(y_1^n)) \\ &\leq 1 - T\{m_A^3(y_1), ..., m_A^3(y_n)\} \\ &= S\{m_A^3(y_1), m_A^3(y_2), ..., m_A^3(y_{1-n})\}\end{aligned}$$

And

$$\bar{m}_A^3(\bar{y}) = 1 - m_A^3(y) \leq 1 - m_A^3(y) = m_A^3(\bar{y})$$

Hence, the notion $\diamondsuit A$ is an extension of fermatean uncertainty of G, ρ. □

Definition 3.3. Let $A = (m_A^3, n_A^3)$ be a fermatean uncertainty collection in G.
Let $k \in [0, 1]$ with the bounds of membership function on G is defined by

$$U(m_A^3; k) = \{y \in G/m_A^3(y) \geq k\},$$
$$L(n_A^3; k) = \{y \in G/n_A^3(y) \leq k\}$$

Here in, the symbol m_A^3 and n_A^3 denotes the $levelk - cut$ in G.
The consequence of the following theorems to be proved based on the results discussed in Dudek.

Theorem 3.3. *Let A be the fermatean uncertainty collection in G with the image $Im(m_A^3) = \{k_i/i \in I\}$ and $Im(n_A^3) = k_j/j \in I$ is also an extension fermatean uncertainty subgroup of (G, ρ). If and only if the m_A^3-level k-cut and n_A^3-level k-cut of G are $k \in Im(m_A^3) \cap Im(n_A^3)$, which are known as m_A^3-level extension subgroups and n_A^3-level extension subgroups respectively.*

Proof. Let A be an extension fermatean uncertainty subgroup of (G, ρ). If $y_1 \in G$ and $k \in [0, 1]$, then $m_A^3(y_1) \geq k$ for all $i = 1, 2, 3..., n$ thus

$$m_A^3(\rho(y_1^n)) \geq T\{m_A^3(y_1), ..., m_A^3(y_n)\} \geq k$$

this implies $(\rho(y_1^n)) \in U(m_A^3; k)$ and $(\rho(y_1^n)) \in U(m_A^3; k)$ and
$n_A^3(\rho(y_1^n)) \leq Sn_A^3(y_1), ..., m_A^3(y_n) \leq k$
implies $(\rho(y_1^2)) \in L(n_A^3; k)$.
Moreover, for some$y \in U(m_A^3; k)$ and $y \in L(n_A^3; k)$ we have
$m_A^3(\bar{y}_1) \geq m_A^3(y_1) \geq k$ and $n_A^3(\bar{y}_1) \leq n_A^3(y_1) \leq k$
implies $y \in U(m_A^3; k)$and$y \in L(n_A^3; k)$.

Thus, m_A^3-level k-cut and n_A^3-level k-cut are also an extension subgroup of (G, ρ).
Conversely, assume that both m_A^3-level k-cut and n_A^3-level k-cut are extension subgroup

of (G, ρ).

Define 1

$k_0 = T\{m_A^3(y_1), ..., m_A^3(y_n)\}$ and $k_1 = T\{n_A^3(y_1), ..., n_A^3(y_n)\}$

For some $y_1^n \in\ G$ then $y_1^n \in U(m_A^3; k_0)$ and $y_1^n \in L(n_A^3; k_0)$

Consequently $\rho(y_1^n) \in U(m_A^3; k_0)$ and $\rho(y_1^n) \in L(n_A^3; k_0)$

Thus, $m_A^3(\rho(y_1^n)) \geq k_0 = Tm_A^3(y_1), ..., m_A^3(y_n)$

$n_A^3(\rho(y_1^n)) \leq k_1 = Sn_A^3(y_1), ..., m_A^3(y_n).$

$Taking\ y_1^n \in U(m_A^3; k)$ and $y_1^n \in L(n_A^3; k) then$

$m_A^3(y) = k_0 \geq k$ and $n_A^3(y) = k_1 \leq k$

Thus, $y \in U(m_A^3; k_0)$ and $y \in L(n_A^3; k_1)$.

Since, by the assumption, $\bar{y} \in U(m_A^3; k_0)$ and $\bar{y} \in U(m_A^3; k_0)$,

Where $m_A^3(\bar{y}) \geq k_0 = m_A^3(y)$ and $n_A^3(\bar{y}) \geq k_1 = n_A^3(y)$.

This completes the proof. □

Theorem 3.4. *Let A be the fermatean uncertainty collection in G is an extension fermatean uncertainty subgroup of (G,p) if and only if m_A^3 - level k-cut and n_A^3 - level k-cut of G are also extension subgroup of (G,p). For each i=1,2,3,$\cdots$,n and $y_1^n \in G$ satisfies the following conditions:*

(i) $m_A^3\big(\rho(y_1^n)\big) \geq T\Bigg\{m_A^3\big(y_1\big), \cdots, m_A^3\big(y_n\big)\Bigg\}$

(ii) $n_A^3\big(\rho(y_1^n)\big) \leq S\Bigg\{n_A^3\big(y_1\big), \cdots, n_A^3\big(y_n\big)\Bigg\}$

(iii) $m_A^3\big(y_1^n\big) \geq T\Bigg\{m_A^3\big(y_1\big), \cdots, m_A^3\big(y_{i-1}\big), m_A^3\big(\rho\big(y_{1j}\big)^n\big), m_A^3\big(y_{i-1}\big)m_A^3\big(y_n\big)\Bigg\}$

(iii) $n_A^3\big(y_1^n\big) \leq S\Bigg\{n_A^3\big(y_1\big), \cdots, n_A^3\big(y_{i-1}\big), n_A^3\big(\rho\big(y_{1j}\big)^n\big), n_A^3\big(y_{i-1}\big)n_A^3\big(y_n\big)\Bigg\}.$

Proof. According to Theorem 3.3, for each non -empty level subsets both $U\Bigg(m_A^3; k_0\Bigg)$ and $L\Bigg(n_A^3; k_1\Bigg)$ are closed under ρ operation in $\big(G, \rho\big)$ with $y_1^n \in U\Bigg(m_A^3; k_0\Bigg)$ and $y_1^n \in L\Bigg(n_A^3; k_1\Bigg)$ satisfies $\rho\big(y_1^n\big) \in U\Bigg(n_A^3; k_1\Bigg)$ and $\rho\big(y_1^n\big) \in L\Bigg(n_A^3; k_1\Bigg)$.

For some i=1,2,$\cdots$,n and $z \in G$ we have $y_0, y_i^{i-1}, y_{i+1}^n$, where $\rho\Bigg(y_i^{i-1}, z, y_{i+1}^n\Bigg)$ satisfies $y_0 \in U\Bigg(m_A^3; k_0\Bigg)$ and $y_0 \in L\Bigg(m_A^3; k_1\Bigg)$. Therefore, $z \in m_A^3(k)$ and $z \in n_A^3(z)$ is a solution of Eqn.(1). Conclude that both level k-cuts m_A^3 and n_A^3 is also a extension subgroups in G.

Conversely, assume that bothe level k-cuts m_A^3 and n_A^3 in G be an extension subgroups then the following conditions in (i) and (ii) are proved.

For $y_1^n \in G$, we consider

$$k_0 = T\Big\{m_A^3(y_1), \cdots, m_A^3(y_{i-1}), m_A^3(\rho(y_{1j})^n), m_A^3(y_{i-1})m_A^3(y_n)\Big\}$$

$$k_1 = S\Big\{n_A^3(y_1), \cdots, n_A^3(y_{i-1}), n_A^3(\rho(y_{1j})^n), n_A^3(y_{i-1})n_A^3(y_n)\Big\}$$

then it satisfies $y_i^{i-1}, y_{i+1}^n, \rho\Big(y_1^n\Big) \in U\Big(m_A^3; k_0\Big)$ and $y_i^{i-1}, y_{i+1}^n, \rho(y_1^n) \in L(n_A^3; k_0)$.

According to Definition 3.1: then we obtained as $y_i \in U(m_A^3; k_0)$ and $y_i \in L(n_A^3; k_0)$. Thus, $m_A^3(y_i) \le k_0$ and $n_A^3(y_i) \le k_1$ satisfies the conditions (iii) and (iv). □

Definition 3.4. The Subgroup (G, ρ) and (G^1, ρ) are under fermatean extension and for any $y_i^n \in G$ define a homomorphism map: $\alpha : G \to G^1$ defined by $\alpha(\rho(y_1^n)) = \rho(\alpha^n(y_1^n))$ where $\alpha^n(y_1^n) = \alpha(y_1), \cdots, \alpha(y_n)$.
The above homomorphism map is said to be a fermatean extension homomorphism. For any fermatean uncertainty collection in G defined by

$$\alpha^{-1}(A) = \Big(m_A^3\alpha^{-1}(A), n_A^3\alpha^{-1}(A)\Big),$$

where, $m_A^3\alpha^{-1}(A)(y) = m_A^3(\alpha(y))$ and $n_A^3\alpha^{-1}(A)(y) = n_A^3(\alpha(y))\forall y \in G$.
For any $A \in G$, the symbol $\alpha(A)$ as the image of A under α is also a fermatean uncertainty collection in G^1 defined by $\alpha(A) = (\alpha_{sup}(m_A^3), \alpha_{inf}(n_A^3))$.
For all $y \in G$ and $x \in G^1$ then

$$\alpha_{sup}(x) = \begin{cases} \sup\limits_{y\in\alpha^{-1}(x)} m_A^3(y), & \text{if } \alpha^{-1}(x) \neq \phi, \\ 0, & \text{Otherwise} \end{cases}$$

$$\alpha_{inf}(x) = \begin{cases} \inf\limits_{y\in\alpha^{-1}(x)} n_A^3(y), & \text{if } \alpha^{-1}(x) \neq \phi, \\ 0, & \text{elsewhere} \end{cases}$$

Theorem 3.5. *For all $y \in G$, an into map $\alpha : G \to G^1$ is defined by $\alpha(\bar{y}) = \alpha(y)$. If A be an extension fermatean uncertainty subgroup of (G^1, ρ) then $\alpha^{-1}(A)$ is also an extension fermatean uncertainty subgroup under (G, ρ).*

Proof. Let $y_1^n \in G$ We have,

$$\begin{aligned} m_{\alpha^{-1}(A)}^3(\rho(y_1^n)) =& m_A^3(\alpha(y_1^n) \\ =& m_A^3(\alpha\rho(\alpha^n)(y_1^n)) \\ \ge& T\{m_A^3(\alpha(y_1)), ..., m_A^3(\alpha(y_n))\} \\ =& T\{m_{\alpha^{-1}(A)}^3(y_1), ..., m_{\alpha^{-1}(A)}^3(y_n)\} \\ n_{\alpha^{-1}(A)}^3(\rho(y_1^n)) =& n_A^3(\alpha(y_1^n)) \\ =& n_A^3(\alpha\rho(\alpha^n)(y_1^n)) \\ \ge& S\{n_A^3(\alpha(y_1)), .., n_A^3(\alpha(y_n))\} \\ =& S\{n_{\alpha^{-1}(A)}^3(y_1), ..., n_{\alpha^{-1}(A)}^3(y_n)\} \\ m_{\alpha^{-1}(A)}^3(\bar{y}) = m_A^3(\alpha(\bar{y})) \ge& m_A^3(\alpha(y)) = m_{\alpha^{-1}(A)}^3(y) \\ n_{\alpha^{-1}(A)}^3(\bar{y}) = n_A^3(\alpha(\bar{y})) \le& n_A^3(\alpha(y)) = n_{\alpha^{-1}(A)}^3(y) \end{aligned}$$

Converse part is obviously true to be discussed in Theorem 3.10. □

Theorem 3.6. *Let $\alpha : G \to G^1$ be an extension homomorphism. If $\alpha^{-1}(A)$ is an extension fermatean uncertainty subgroup of (G, ρ) then A is also an extension fermatean uncertainty group of (G^1, ρ).*

Theorem 3.7. *Let $\alpha : G \to G^1$ be a map. If A is an extension fermatean uncertainty subgroup of (G, ρ) then $\alpha(A) = (y, \alpha_{sup}(m_A^3), \alpha_{inf}(n_A^3))$ is also an extension fermatean uncertainty subgroup of (G^1, ρ) .*

Proof. Let $\alpha : G \to G^1$ be a map and for any $x_1^n, y_1^n \in$ G then

$\{y_i/y_i \in \alpha^{-1}(\rho(x_1^n))\} \subset \{\rho(y_1^n) \in G/x_1 \in \alpha^{-1}(x_1), y_2 \in \alpha^{-1}(x_2), ..., y_n \in \alpha^{-1}(x_n)\}$

We have $\alpha_{sup}(m_A^3)(\rho(x_1^n)) = sup\{m_A^3(y_1^n)/y_i \in \alpha^{-1}(\rho(x_1^n))\}$

$\geq sup\{m_A^3(\rho(y_1^n))/y_1 \in \alpha^{-1}(x_1), y_2 \in \alpha^{-1}(x_2), ..., y_n \in \alpha^{-1}(x_n)\}$

$\geq sup\ \{T\{m_A^3(y_1), m_A^3(y_2), ..., m_A^3(y_n)/y_1 \in \alpha^{-1}(x_1), y_2 \in \alpha^{-1}(x_2), ..., y_n \in \alpha^{-1}(x_n)\}\}$

$= T\{sup\{m_A^3(y_1)/y_1 \in \alpha^{-1}(x_1)\}, sup\{m_A^3(y_2)/y_2 \in \alpha^{-1}(x_1)\}, ..., sup\{m_A^3(y_n) /y_1 \in \alpha^{-1}(x_n)\}$

$= T\{\alpha_{sup}(m_A^3(x_1)), \alpha_{sup}(m_A^3(x_2)), ..., \alpha_{sup}(m_A^3(x_n))\}$

$\alpha_{sup}(n_A^3)(\rho(x_1^n)) = inf\{n^3A(y_1^n)/y_1 \in \alpha^{-1}(\rho(x_1^n))\}$

$\leq inf\{n_A^3(\rho(y_1^n))/y_1 \in \alpha^{-1}(x_1), y_2 \in \alpha^{-1}(x_2), ..., y_n \in \alpha^{-1}(x_n)\}$

$\leq inf\{S\{n_A^3(y_1), n_A^3(y_2), ...n_A^3(y_n)/y_1 \in \alpha^{-1}(x_1), y_2 \in \alpha^{-1}(x_2), ..., y_n \in \alpha^{-1}(x_n)\}\}$

$= S\{inf\{n_A^3(y_1)/y_1 \in \alpha^{-1}(x_1)\}, inf\{n_A^3(y_)/y_2 \in \alpha^{-1}(x_2)\}, ..., inf\{n_A^3(y_n)/y_n \in \alpha^{-1}(x_n)\}\}$

$= S\{\alpha_{inf}(n_A^3(x_1)), \alpha_{inf}(n_A^3(x_2)), ..., \alpha_{inf}(n_A^3(x_n))\}$

$\alpha_{sup}(m_A^3(y)) = sup\{m_A^3(y)/y \in \alpha^{-1}(\rho(x))\}$

$\geq sup\{m_A^3(y)/y \in \alpha^{-1}(\rho(x))\}$

$= \alpha_{sup}(m_A^3)(y))$

$\alpha_{sup}(n_A^3)(y) = \inf\{(n_A^3)(y)/y \in \alpha^{-1}(\rho(x))\}$

$\leq \inf\{n_A^3(y)/y \in \alpha^{-1}(\rho(x))\} \leq \inf\{n_A^3(y)/y \in \alpha^{-1}(\rho(x))\} = \alpha_{sup}(n_A^3)(y).$

Hence the theorem is proved. □

Corollary 3.8. *A fermatean uncertainty collection A on (G,•) is a fermatean uncertainty subgroup if and only if it satisfies the following relations for any x,y ∈ G given by*

(i) $m_A^3(y) \geq T\{m_A^3(x), m_A^3(y)\}$ *and* $n_A^3(xy) \leq S\{n_A^3(x),\ n_A^3(y)\}$

(ii) $m_A^3(x) \geq T\{m_A^3(y), m_A^3(xy)\}$ *and* $n_A^3(x) \leq S\{n_A^3(y),\ n_A^3(xy)\}$

(iii) $m_A^3(y) \geq T\{m_A^3(x), m_A^3(xy)\}$ *and* $n_A^3(y) \leq S\{n_A^3(x),\ n_A^3(xy)\}$

Theorem 3.9. *Let A be a collection of an extension fermatean uncertainty subgroup (G, ρ). For any $x, y \in G$, there exist $a \in G$ such that $m_A^3(a) \geq m_A^3(x)$ and $n_A^3(a) \leq n_A^3(x)$ then A is an extension fermatean uncertainty subgroup of $ret_\alpha(G, \rho)$.*

Proof. For each $a, x, y \in G$, we have the relation

$$\begin{aligned} m_A^3(xy) &= m_A^3(\rho(x, a^{(n-2)}, y)) \\ &\geq T\{m_A^3(x), m_A^3(a), m_A^3(y)\} \\ &= T\{m_A^3(x), m_A^3(y)\} \\ \text{Also, we write} \quad n_A^3(xy) &= n_A^3(\rho(x, a^{(n-2)}, y)) \\ &\leq S\{n_A^3(x), n_A^3(a), n_A^3(y)\} \\ &= S\{n_A^3(x), n_A^3(y)\} \end{aligned}$$

$$\begin{aligned} m_A^3(x^{-1}) = m_A^3(\rho(a, x^{(n-3)}, \bar{x}, \bar{a})) &\geq T\{m_A^3(x), m_A^3(\bar{x}), m_A^3(a), m_A^3(\bar{a})\} \\ = m_A^3(x) & \\ n_A^3(x^{-1}) = n_A^3(\rho(a, x^{(n-3)}, \bar{x}, \bar{a})) &\geq S\{n_A^3(x), n_A^3(\bar{x}), n_A^3(a), n_A^3(\bar{a})\} \\ = n_A^3(x) & \end{aligned}$$

In theorem 3.9, for that we have $m_A^3(a) \geq m_A^3(x)$ and $n_A^3(a) \leq n_A^3(x)$

This completes the proof. Followed by the numerical example is discussed here. □

Example 3.5. Define a map: $\rho : Z_4^3 \to Z_4$ on (Z_4, ρ) defined by $\rho(y_1, y_2, y_3) = S(y_1, y_2, y_3)$. Clearly the ternary subgroup (Z_4, ρ) obtained from Z_4 under this map. The fermatean uncertainty collection A is given by

$$m_A^3 = \begin{cases} 1, & if\ y = 0, \\ 0.3, & if\ y = 1, 2, 3 \end{cases}$$

$$n_A^3 = \begin{cases} 0, & if\ y = 0, \\ 0.8, & if\ y = 1, 2, 3 \end{cases}$$

Clearly, we say that A is also a fermatean uncertainty ternary subgroup of (Z_4, ρ) For $ret_(Z_4, \rho)$. we have the algebraic operations are
$m_A^3(0 \bullet 0) = m_A^3(\rho(0, 1, 0)) = m_A^3(1) = 0.3$ is not greater than $T\{m_A^3(0), m_A^3(0)\} = 1$.
Also, $n_A^3(0 \bullet 0) = n_A^3(\rho(0, 1, 0)) = n_A^3(1) = 0.8$ is not less than $S\{n_A^3(0), n_A^3(0)\} = 1$.
Hence, for that we have $m_A^3(a) \geq m_A^3(x)$ and $n_A^3(a) \leq n_A^3(x)$.

Theorem 3.10. *Let A be an extensiion fermatean uncertainty subgroup of* $\textbf{ret}_\alpha(G, \rho)$*. For any $x, a \in G$ satisfies $m_A^3(a) \geq m_A^3(x)$ and $n_A^3(a) \leq n_A^3(x)$ then A is an extension fermatean uncertainity subgroup under (G, ρ).*

Proof. From equation 2.1 for any extension subgroup of the form given in Eqn.(2), we have

$$(G, \cdot) = \text{ret}_\alpha(G, \rho), \phi(x) = \rho(a, y, y^{(n-2)}) \quad \text{and} \quad b = \rho(\bar{a}, ..\bar{a}) \quad \text{then}$$

$$\begin{aligned}
m_A^3(\phi(y)) &= m_A^3(\rho(\bar{a}, y, y^{(n-2)})) \\
&\geq T\{m_A^3(\bar{a}), m_A^3(y), m_A^3(a)\} = m_A^3(y), \\
m_A^3(\phi^2(y)) &= m_A^3(\rho(\bar{a}, \phi(y), y^{(n-2)})) \\
&\geq T\{m_A^3(\bar{a}), m_A^3(\phi(y)), m_A^3(a)\} = m_A^3(\phi(y)), \\
&\geq m_A^3(y)
\end{aligned}$$

In general, for any $y \in G, K \in N$ then the relation $m_A^3(\phi^k(y)) \geq m_A^3(y)$ is true.
Similarly,

$$\begin{aligned}
n_A^3(\phi(y)) &= n_A^3(\rho(\bar{a}, y, y^{(n-2)})) \\
&\geq S\{n_A^3(\bar{a}), n_A^3(y), n_A^3(a)\} = n_A^3(y), \\
n_A^3(\phi^2(y)) &= n_A^3(\rho(\bar{a}, \phi(y), y^{(n-2)})) \\
&\geq S\{n_A^3(\bar{a}), n_A^3(\phi(y)), n_A^3(a)\} = n_A^3(\phi(y)), \\
&\geq n_A^3(y)
\end{aligned}$$

In general, for any $y \in G, K \in N$ then the relation $n_A^3(\phi^k(y)) \geq n_A^3(y)$ is true.
For every $x \in G$, the following relations are obviously true. We have,

$$m_A^3(b) = m_A^3(\rho(\bar{a}, ...\bar{a})) \geq m_A^3(\bar{a}) \geq m_A^3(y)$$
$$n_A^3(b) = n_A^3(\rho(\bar{a}, ...\bar{a})) \geq n_A^3(\bar{a}) \geq n_A^3(y)$$

thus,

$$\begin{aligned}
m_A^3(\rho(y_1)) &= m_A^3(y_1 \circ \phi(y_2) \circ \phi^2(y_3) \circ ... \circ \phi^{n-2}(y_n) \circ b) \\
&\geq T_1\{m_A^3(\phi(y_1)), m_A^3(\phi(y_2)), ...m_A^3(\phi(y_3)), ..., m_A^3(\phi(y_n)), m_A^3(b)\} \\
&\geq T\{m_A^3(y_1), m_A^3(y_2), m_A^3(y_3), ..., m_A^3(y_n), m_A^3(b)\} \\
&\geq T\{m_A^3(y_1), m_A^3(y_2), m_A^3(y_3), ..., m_A^3(y_n), \} \quad \text{and} \\
n_A^3(\rho(y_1)) &= n_A^3(y_1 \circ \phi(y_2) \circ \phi^2(y_3) \circ ... \circ \phi^{n-2}(y_n) \circ b) \\
&\geq S_1\{n_A^3(\phi(y_1)), n_A^3(\phi(y_2)), ...n_A^3(\phi(y_3)), ..., n_A^3(\phi(y_n)), n_A^3(b)\} \\
&\geq S\{n_A^3(y_1), n_A^3(y_2), n_A^3(y_3), ..., n_A^3(y_n), n_A^3(b)\} \\
&\geq S\{n_A^3(y_1), n_A^3(y_2), n_A^3(y_3), ..., n_A^3(y_n), \}
\end{aligned}$$

Finally, we have,

$$\bar{y} = (\phi(y) \circ \phi^2(y) \circ .... \circ \phi^{n-2}(y) \circ b)^{-1}$$

Thus,

$$\begin{aligned} m_A^3(y) &= m_A^3(\phi(y) \circ \phi^2(y) \circ .... \circ \phi^{n-2}(y) \circ b)^{-1} \\ &\geq m_A^3(\phi(y) \circ \phi^2(y) \circ .... \circ \phi^{n-2}(y) \circ b) \\ &\geq T\{m_A^3(\phi(y)) \circ m_A^3\phi^2(y) \circ m_A^3\phi^{n-2}(y) \circ m_A^3(b)\} \\ &\geq T\{m_A^3(y) \circ \geq T\{m_A^3(b)\} = m_A^3 \quad \text{and} \\ n_A^3(y) &= n_A^3(\phi(y) \circ \phi^2(y) \circ .... \circ \phi^{n-2}(y) \circ b)^{-1} \\ &\geq n_A^3(\phi(y) \circ \phi^2(y) \circ .... \circ \phi^{n-2}(y) \circ b) \\ &\geq S\{n_A^3(\phi(y)) \circ n_A^3\phi^2(y) \circ n_A^3\phi^{n-2}(y) \circ n_A^3(b)\} \\ &\geq S\{n_A^3(y) \circ \geq T\{n_A^3(b)\} = n_A^3. \end{aligned}$$

This result is derived. □

Corollary 3.11. *Let (G, ρ) be a ternary group. For any fermatean uncertainty subgroup $ret_\alpha(G, \rho)$ is also a fermatean uncertainty ternary subgroup under (G, ρ).*

Proof. Let $\overline{a}$ be an intermediate element of $ret_\alpha(G, \rho)$. For each $y \in G$, the following inequalities are $m_A^3(a) \leq m_A^3(\overline{y})$ and $n_A^3(a) \leq n_A^3(\overline{y})$.Then we have $m_A^3(\overline{a}) \leq m_A^3(y)$ and $n_A^3(\overline{a}) \leq n_A^3(y)$. But in ternary group $\overline{a} = a$ (involution operator). For any $a, y \in G$ we have $m_A^3(a) = m_A^3(\overline{a}) \geq m_A^3(\overline{a}) \geq m_A^3(y)$ and $n_A^3(a) = n_A^3(\overline{a}) \leq n_A^3(\overline{a}) \leq m_A^3(y)$. So $m_A^3(a) = m_A^3(\overline{a}) \geq m_A^3(y)$ and $n_A^3(a) = n_A^3(\overline{a}) \leq n_A^3(y)$ The above relations are satisfying the Theorem 3.10. Followed by the numerical example is discussed here. □

Example 3.6. Define a map $\rho : Z_{12}^3 \to Z_{12}$ under the additive group (Z_{12}, ρ) defined by $\rho(y_1, y_2, y_3) = S(y_1, y_2, y_3)$. Let A be a fermatean uncertainty subgroup of $ret_t(G, \rho)$ induced by subgroups $S_1 = \{11\}$, $S_2 = \{5, 11\}$ and $S_3 = \{1, 3, 5, 7, 9, 11\}$
Define the fermatean uncertainty collection as follows

$$m_A^3(y) = \begin{cases} 0.7 & \text{if } \; y = 1; \\ 0.5 & \text{if } y = 5; \\ 0.3 & \text{if } y = 1, 3, 7, 9; \\ 0.1 & \text{if } \; y \in S_3; \end{cases}$$

$$n_A^3(y) = \begin{cases} 0.2 & \text{if } \; y = 11; \\ 0.4 & \text{if } y = 5; \\ 0.6 & \text{if } y = 1, 3, 7, 9; \\ 0.8 & \text{if } \; y \in S_3; \end{cases}$$

Then, $m_A^3(5) = m_A^3(7) = 0.3$ is not greater than are equal to $0.5 = m_A^3(5)$
$n_A^3(5) = n_A^3(7) = 0.6$ is not less than are equal to $0.4 = n_A^3(5)$
Hence, the collection A is not a fermatean uncertainty ternary subgroup.

Remark. *From Example 3.6, the following results are true.*

1 *The fermatean uncertainty subgroups of group $ret_\alpha(G, \rho)$ are not an extension fermatean uncertainty subgroups.*
2 *In Theorem 3.10 the conditions $m_A^3(y) \geq m_A^3(y)$ and $n_A^3(y) \leq n_A^3(y)$ cannot be neglected. Also, in the Example 3.6 we have $m_A^3(a) = 0.3$ is not greater than are equal to $0.5 = m_A^3(5)$ and $n_A^3(5) = 0.6$ is not less than are equal to $0.4 = n_A^3(5)$.*

3 The relations $m_A^3(a) \geq m_A^3(y)$ and $n_A^3(a) \leq n_A^3(y)$ cannot be replaced by $m_A^3(\overline{a}) \geq m_A^3(y)$ and $n_A^3(\overline{a}) \leq n_A^3(y)$, where $\overline{a}$ is an identity element of $ret_\alpha(G,\rho)$. In Example 3.6, for any $y \in Z_{12}$, $T = 11$ then $m_A^3(11) \geq m_A^3(y)$ and $n_A^3(11) \leq n_A^3(y)$.

Theorem 3.12. *An extension subgroup (G,ρ) of b-derived group $(G,\bullet)$ For any collection in an extension fermatean uncertainty subgroup under$(G,\bullet)$ such that $m_A(b) \geq m_A(y)$ and $n_A(b)leqn_A(y)$ $\forall y \in G$ is also an extension fermatean uncertainty subgroup under(G,ρ)*

Proof. According to the extension principle rules $(EFUSG_1)$ and $(EFUSG_2)$ defined in Definition 3.1: to prove the extension principle rules $(EFUSG_3)$ and $(EFUSG_4)$ under the extension group $(G,\bullet)$, the b-exempted from an extension fermatean uncertainty subgroup $(G,\bullet)$ gives $y = (y^{(n-2)} \bullet b)^{-1}$ Followed by for each $y \in G$,

$$\begin{aligned} m_A^3(y) &= m_A^3(y^{(n-2)} \bullet b)^{-1} \\ &\geq m_A^3(y^{(n-2)} \bullet b) \\ &\leq T\{m_A^3(y^{(n-2)}, m_A^3(b))\} \\ &= m_A^3(y) \\ n_A^3(y) &= n_A^3(y^{(n-2)} \bullet b)^{-1} \\ &\geq n_A^3(y^{(n-2)} \bullet b) \\ &\leq S\{n_A^3(y^{(n-2)}, n_A^3(b))\} \\ &= n_A^3(y) \end{aligned}$$

This completes the proof of the result. □

Corollary 3.13. *For any fermatean uncertainty group of a group $(G,\bullet)$ is an extension fermatean uncertainty subgroup of (G,ρ) obtained by $(G,\bullet)$.*

Proof. An extension group (G,ρ) is obtained from $(G,\bullet)$ by substituting an element $b = e$ in Theorem 3.12 gives the relations
$m_A(e) \geq m_A(y), n_A(e) \leq n_A(y), \forall y \in G$.
Therefore, $m_A^3(5) = m_A^3(7) = 0.3$ is not greater than are equal to $0.5 = m_A^3(5)$
$n_A^3(5) = n_A^3(7) = 0.6$ is not less than are equal to $0.4 = n_A^3(5)$
Hence, the collection A is not a fermatean uncertainty ternary subgroup of (G,ρ). □

4. Conclusions

We explained different aspects of cubic fermatean uncertainty soft ideal structures and fermatean uncertainty multi-group over multi-homomorphism. An extension of this fermatean uncertainty study using soft subgroup structures with norm. Moreover, we discussed the important mathematical proofs as Theorems, corollaries and suitable examples. In future, we extended this research into field of picture uncertainty collection and spherical uncertainty collections.

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