



GALOIS CONNECTIONS AND ISOMORPHISM OF SIMULTANEOUS ORDERED RELATIONS

OMPRAKASH ATALE

ABSTRACT. In order theory, partially ordered sets are only equipped with one relation which decides the entire structure/Hasse diagram of the set. In this paper we have presented how partially ordered sets can be studied under simultaneous partially ordered relations which we have called binary posets. The paper is motivated by the problem of operating a set simultaneously under two distinct partially ordered relations. It has been shown that binary posets follow duality principle just like posets do. Within this framework, some new definitions concerning maximal and minimal elements are also presented. Furthermore, some theorems on order isomorphism and Galois connections are derived.

1. INTRODUCTION

Let P be a non-empty set. A relation \preceq on P is known as partially ordered relation (POR) if \preceq is [1] :-

- (1) Reflexive: $a \preceq a, \forall a \in P$.
- (2) Anti-symmetric: $[(a \preceq b) \wedge (b \preceq a)] \implies a = b, \forall a, b \in P$.
- (3) Transitive: $[(a \preceq b) \wedge (b \preceq c)] \implies a \preceq c, \forall a, b, c \in P$.

In order theory, if \preceq is a partially ordered relation on a non-empty set P , then an ordered pair (P, \preceq) is known as a partially ordered set or poset. In this paper, we are going to construct the theory of posets equipped with binary relation $\diamond = (\preceq_1, \preceq_2)$. The partially ordered relations \preceq_1 and \preceq_2 may or may not be same. The reason for constructing the concept of a poset with binary relations is that it would help us to study a partially ordered set with two partially ordered relations simultaneously. Generally, when we study partially ordered sets, we take into account 3 properties: reflexivity, anti-symmetric and transitive property. Under the definition of this properties, we show under what conditions involving a single relation \preceq can two different elements can be related to itself, equal to each-other and related to other elements. The drawback however, is that this only helps us studying non-sets under one relation at a time.

2010 *Mathematics Subject Classification.* 06Fxx, 06Axx, 06A15.

Key words and phrases. Posets; isomorphism; Galois connections.

Received: April 11, 2023. Accepted: May 25, 2023. Published: June 30, 2023.

2. BINARY POSETS

In this paper, we are going to modify the definition of properties of partially ordered relation so that we could study non-empty sets under two relations simultaneously at a time. We call this newly constructed poset a binary poset. It is shown that binary posets satisfy all the properties that a posets does. We can define isomorphism from one binary poset to another binary poset using the definition of a isotone, the binary posets also follow duality principle. The derived theorems and definitions are then applied to derive some results on the power set. In the last section, we have given how a Galois connection between two binary posets can be defined. Some related examples and theorems on Galois connections of binary posets are derived. Throughout the paper \wedge and \vee are short hand mathematical notations for "and" and "or" respectively.

Definition 1 (partially ordered binary relation). *Let P be a non-empty set. A relation $\diamond = (\preceq_1, \preceq_2)$ is known as partially ordered binary relation (POBR) if \diamond is:-*

(1) *Reflexive: $a \preceq_1 a \preceq_2 a, \forall a \in P$.*

(2) *Anti-symmetric: $\forall a, b, c \in P$*

$$[(a \preceq_1 b \preceq_2 c) \wedge (b \preceq_1 a \preceq_2 c) \wedge (a \preceq_1 c \preceq_2 b)] \Rightarrow (a = b = c).$$

(3) *Transitive: $\forall a, b, c, d, e \in P$*

$$[(a \preceq_1 b \preceq_2 c) \wedge (b \preceq_1 d \preceq_2 c) \wedge (c \preceq_2 e)] \Rightarrow [(a \preceq_1 d \preceq_2 c) \wedge (a \preceq_1 b \preceq_2 e)].$$

Above properties are the extension of the binary relations from one that involves one operation to two operations. In the similar manner as above, one can define binary relations for other properties that include symmetric, connected, well-founded, irreflexive and asymmetric properties of sets. Since, in this paper we are only going to deal with partial ordered sets, we won't be needing them here. But in the case of reflexive, anti-symmetric and transitive properties, the definition provided in Def. 1 can be thought of as universal relations operating on two partially ordered relations simultaneously.

Definition 2 (Binary partially ordered set). *Let $\diamond = (\preceq_1, \preceq_2)$ be partially ordered binary relation on a non-empty set P , then an ordered pair (P, \diamond) is known as binary partially ordered set (binary poset).*

The set of all natural numbers under relation $\diamond = (\leq, |)$ forms a binary poset.

Theorem 1. *If (P, \diamond') and $(P, \tilde{\diamond})$ are two binary posets. Then $(P, \diamond' \cap \tilde{\diamond})$ is also a binary poset, where $\diamond' = (\preceq_1, \preceq_2)$ and $\tilde{\diamond} = (\preceq_3, \preceq_4)$.*

Proof: Suppose that (P, \diamond') and $(P, \tilde{\diamond})$ are two binary posets. Let

$$a \preceq' b \preceq'' c \iff [(a \preceq_1 b \preceq_2 c) \wedge (a \preceq_3 b \preceq_4 c)] \quad (2.1)$$

i.e. $\diamond = \diamond' \cap \tilde{\diamond}$ where $a, b, c \in P$. First, we prove reflexive property. Let $a \in P$. Since \diamond' and $\tilde{\diamond}$ are reflexive, we get

$$[(a \preceq_1 a \preceq_2 a) \wedge (a \preceq_3 a \preceq_4 a)] \implies (a \preceq' a \preceq'' a). \quad (2.2)$$

Hence, \diamond is reflexive. Now, we prove anti-symmetric property. Let $a, b, c \in P$. Since \diamond' and $\tilde{\diamond}$ are anti symmetric, we get

$$[(a \preceq_1 b \preceq_2 c) \wedge (b \preceq_1 a \preceq_2 c) \wedge (a \preceq_1 c \preceq_2 b)], \quad (2.3)$$

$$[(a \preceq_3 b \preceq_4 c) \wedge (b \preceq_3 a \preceq_4 c) \wedge (a \preceq_3 c \preceq_4 b)]. \quad (2.4)$$

Thus

$$[(a \preceq' b \preceq'' c) \wedge (b \preceq' a \preceq'' c) \wedge (a \preceq' c \preceq'' b)] \implies (a = b = c). \quad (2.5)$$

Hence, \diamond is anti-symmetric. Now, we prove transitive property. Let $a, b, c, d, e \in P$. Since \diamond' and $\tilde{\diamond}$ are transitive, we get

$$[(a \preceq_1 b \preceq_2 c) \wedge (b \preceq_1 d \preceq_2 c) \wedge (c \preceq_2 e)] \implies [(a \preceq_1 d \preceq_2 c) \wedge (a \preceq_1 b \preceq_2 e)], \quad (2.6)$$

$$[(a \preceq_3 b \preceq_4 c) \wedge (b \preceq_3 d \preceq_4 c) \wedge (c \preceq_4 e)] \implies [(a \preceq_3 d \preceq_4 c) \wedge (a \preceq_3 b \preceq_4 e)]. \quad (2.7)$$

Thus,

$$[(a \preceq' b \preceq'' c) \wedge (b \preceq' d \preceq'' c) \wedge (c \preceq'' e)] \implies [(a \preceq' d \preceq'' c) \wedge (a \preceq' b \preceq'' e)]. \quad (2.8)$$

Hence, \diamond is transitive. Therefore, $(P, \diamond) = (P, \diamond' \cap \tilde{\diamond})$ is a binary poset. The above theorem can be generalized as follows.

Theorem 2. *If (P, \diamond_n) , $n = 0, 1, 2, \dots, m$ are binary posets where $\diamond_n = (\preceq_{n+1}, \preceq_{n+2})$. Then, (P, \diamond) is also a binary poset where $\diamond = \diamond_0 \cap \diamond_1 \cap \diamond_2 \cap \dots \cap \diamond_m$.*

3. MAXIMAL AND MINIMAL ELEMENTS

Let (P, \preceq) be a poset. Then, the notation $a(\forall, \in, \notin)P$ means "for all, or for some, or no" $a \in P$. For example, let $a, b \in P$ and $a \preceq b$. Now, a being fixed, there might be possibility that $a \preceq b$ will be true for all $b \in P$, or for some $b \in P$ or for no $b \in P$. Therefore we collectively write $a \preceq b, b(\forall, \in, \notin)P$

Definition 3 (Minimal and maximal greatest element). *Let (P, \diamond) be a binary poset where $\diamond = (\preceq_1, \preceq_2)$. Let $x, y \in P$ be any two elements such that*

$$[(a \preceq_1 x \preceq_2 c), (\forall a \in P) \wedge c(\forall, \in, \notin)P] \wedge [(a \preceq_1 b \preceq_2 y), (\forall b \in P) \wedge a(\forall, \in, \notin)P]. \quad (3.1)$$

Then, $g_{max} = \sup \{x, y\}$ is known as the maximal greatest element and $g_{min} = \inf \{x, y\}$ is known as the minimal greatest element.

The above definition can be summarized as follows. Lets say we have a non-empty set P equipped with partially ordered binary relation $\diamond = (\preceq_1, \preceq_2)$. Then, the greatest elements generated by \preceq_1 and \preceq_2 may or may not be equal. If they are equal, then we are through. If they are not equal, then the greatest one will be maximal greatest element and the smallest one will be the minimal greatest element. Similarly, for minimal and maximal least element, we have the following definition.

Definition 4 (Minimal and maximal least element). *Let (P, \diamond) be a binary poset where $\diamond = (\preceq_1, \preceq_2)$. let $x, y \in P$ be any two elements such that*

$$[(x \preceq_1 b \preceq_2 c), (\forall b \in P) \wedge c(\forall, \in, \notin)P] \wedge [(a \preceq_1 y \preceq_2 c), (\forall c \in P) \wedge a(\forall, \in, \notin)P]. \quad (3.2)$$

Then, $l_{max} = \sup \{x, y\}$ is known as the maximal least element and $l_{min} = \inf \{x, y\}$ is known as the minimal least element.

Definition 5 (Bounded binary poset). *A binary poset is known as bounded if it has maximal greatest element and minimal least element.*

Definition 6 (Unbounded binary poset). *A binary poset is known as unbounded if it is not a bounded binary poset.*

Theorem 3. *Let (P, \diamond) be a binary poset where $\diamond = (\preceq_1, \preceq_2)$. Then, the maximal greatest element (if it exists) is unique.*

Proof: Suppose that g' and \tilde{g} are the two maximal greatest elements of a binary poset (P, \diamond) where $\diamond = (\preceq_1, \preceq_2)$. Using definition 2, we can write $g' = \sup \{g_1, g_2\}$ and $\tilde{g} = \sup \{g_3, g_4\}$ where $g_1, g_2, g_3, g_4 \in P$ such that

$$[(a \preceq_1 g_1 \preceq_2 c), (\forall a \in P) \wedge c (\forall, \in, \notin) P] \wedge [(a \preceq_1 b \preceq_2 g_2), (\forall b \in P) \wedge a (\forall, \in, \notin) P]. \quad (3.3)$$

$$[(g_3 \preceq_1 g_1 \preceq_2 c), c (\forall, \in, \notin) P] \wedge [(a \preceq_1 g_4 \preceq_2 g_2), a (\forall, \in, \notin) P]. \quad (3.4)$$

Now, $g_3, g_4 \in P$. Therefore,

$$[(g_1 \preceq_1 g_3 \preceq_2 c), c (\forall, \in, \notin) P] \wedge [(a \preceq_1 g_2 \preceq_2 g_4), a (\forall, \in, \notin) P]. \quad (3.5)$$

Also, $g_1, g_2 \in P$. Therefore, Now, $g_3, g_4 \in P$. This implies $g_1 = g_3$ and $g_2 = g_4$. And hence, $g' = \tilde{g}$. This completes our proof.

Similarly, for the uniqueness of minimal greatest element we have the following theorem.

Theorem 4. *Let (P, \diamond) be a binary poset where $\diamond = (\preceq_1, \preceq_2)$. Then, the minimal greatest element (if it exists) is unique.*

The same result applies for the uniqueness of minimal and maximal least element.

Theorem 5. *Let (P, \diamond) be a binary poset where $\diamond = (\preceq_1, \preceq_2)$. Then, the maximal least element (if it exists) is unique.*

Theorem 6. *Let (P, \diamond) be a binary poset where $\diamond = (\preceq_1, \preceq_2)$. Then, the minimal least element (if it exists) is unique.*

Proof of Theorem 5 and 6 follows from the same technique of proof of Theorem 3.

Theorem 7. *Let X be a non-empty set and $P(X)$ be the power set of X . Define $\diamond = (\preceq_1, \preceq_2) = (\subseteq, \subseteq)$ on $P(X)$ by*

$$A \preceq_1 B \preceq_2 C \iff A \subseteq B \subseteq C. \quad (3.6)$$

Then $(P(X), \diamond)$ is a binary poset.

Proof: Let $A, B, C, D, E \in P(X)$. Since every subset is a subset of itself, we can show that $A \subseteq A \subseteq A, \forall A \in P(X)$. Therefore, \diamond is reflexive. Secondly, since

$$\begin{aligned} & [(A \subseteq B \subseteq C) \wedge (B \subseteq A \subseteq C) \wedge (A \subseteq C \subseteq B)] \\ & \implies (A = B = C), \forall A, B, C \in P(X), \end{aligned} \quad (3.7)$$

therefore \diamond is anti-symmetric. Now, we know that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Therefore, if

$$[(A \subseteq B \subseteq C) \wedge (B \subseteq D \subseteq C) \wedge (C \subseteq E)] \implies (A \subseteq D \subseteq C) \wedge (A \subseteq B \subseteq E). \quad (3.8)$$

Hence, \diamond is a partial order binary relation and thus $(P(X), \diamond)$ is a binary poset.

4. BINARY POSET ISOMORPHISM

Let (P, \diamond) and $(Q, \tilde{\diamond})$ be two binary posets where $\diamond = (\preceq_1, \preceq_2)$ and $\tilde{\diamond} = (\preceq_3, \preceq_4)$, A function $\psi : (P, \diamond) \rightarrow (Q, \tilde{\diamond})$ is known as isomorphism if

- (1) For $a, b, c \in P, a \preceq_1 b \preceq_2 c \iff \psi(a) \preceq_3 \psi(b) \preceq_4 \psi(c)$,
- (2) ψ is bijective.

Symbolically, we write $(P, \diamond) \cong (Q, \tilde{\diamond})$

Definition 7 (Isotone). A function $\psi : (P, \diamond) \rightarrow (Q, \diamond)$ where $\diamond = (\preceq_1, \preceq_2)$ is known as isotone if

$$a \preceq_1 b \preceq_2 c \iff \psi(a) \preceq_1 \psi(b) \preceq_2 \psi(c), \forall a, b, c \in P. \quad (4.1)$$

Theorem 8. A function $\psi : (P, \diamond) \rightarrow (Q, \tilde{\diamond})$ is an binary poset isomorphism iff ψ is isotone and has isotone inverse.

Proof: Suppose that $\psi : (P, \diamond) \rightarrow (Q, \diamond)$ is a binary poset isomorphism where $\diamond = (\preceq_1, \preceq_2)$. Therefore, ψ is bijective and satisfies the condition that for $a, b, c \in P$,

$$(a \preceq_1 b \preceq_2 c) \iff \psi(a) \preceq_1 \psi(b) \preceq_2 \psi(c). \quad (4.2)$$

By definition of isotone, ψ is an isotone if it is bijective. Therefore, $\psi^{-1} : (P, \diamond) \rightarrow (Q, \diamond)$ exists. Let $\delta, \epsilon, \zeta \in Q$ such that $\delta \preceq_1 \epsilon \preceq_2 \zeta$. Since ψ is surjective, $\exists a, b, c \in P$ such that $\psi(a) = \delta$ and $\psi(b) = \epsilon, \psi(c) = \zeta$. Therefore $a = \psi^{-1}(\delta), b = \psi^{-1}(\epsilon), c = \psi^{-1}(\zeta)$. Now,

$$\begin{aligned} (\delta \preceq_1 \epsilon \preceq_2 \zeta) &\Rightarrow \psi(a) \preceq_1 \psi(b) \preceq_2 \psi(c) \\ &\Rightarrow (a \preceq_1 b \preceq_2 c) \\ &\Rightarrow \psi^{-1}(\delta) \preceq_1 \psi^{-1}(\epsilon) \preceq_2 \psi^{-1}(\zeta). \end{aligned} \quad (4.3)$$

Therefore, ψ^{-1} is isotone.

Definition 8 (Dual). Let $\diamond = (\preceq_1, \preceq_2)$ be a relation on the set P . Then, the dual of \diamond is denoted by $\tilde{\diamond} = (\tilde{\preceq}_1, \tilde{\preceq}_2)$ and is defined as $a \tilde{\preceq}_1 b \tilde{\preceq}_2 c \iff c \preceq_2 b \preceq_1 a$ or $a \tilde{\preceq}_1 b \tilde{\preceq}_2 c \iff b \preceq_1 a$ and $c \preceq_2 b, \forall a, b, c \in P$.

The two different definitions of dual provided above are equal whereas the second definition is easy to use in proof as compare to first.

Theorem 9 (Duality principle for binary posets). Let (P, \diamond) be a binary poset, then $(P, \tilde{\tilde{\diamond}})$ is also a binary poset.

Proof: Let $\diamond = (\preceq_1, \preceq_2), \tilde{\diamond} = (\tilde{\preceq}_1, \tilde{\preceq}_2)$ and $a, b, c \in P$. We know that $a \preceq_1 a$ and $a \preceq_2 a$, thus $a \tilde{\preceq}_1 a \tilde{\preceq}_2 a$. Hence $\tilde{\diamond}$ is reflexive. Let

$$(a \tilde{\preceq}_1 b \tilde{\preceq}_2 c) \wedge (b \tilde{\preceq}_1 a \tilde{\preceq}_2 c). \quad (4.4)$$

Therefore, Eqn. (4.4) implies

$$(b \preceq_1 a) \wedge (c \preceq_2 b) \wedge (a \preceq_1 b) \wedge (c \preceq_2 a) \implies (a = b). \quad (4.5)$$

Similarly, we can further prove that $a = b = c$. Thus, $\tilde{\diamond}$ is anti-symmetric. Now, let

$$(a \tilde{\preceq}_1 b \tilde{\preceq}_2 c) \wedge (b \tilde{\preceq}_1 d \tilde{\preceq}_2 c) \wedge (c \tilde{\preceq}_1 e). \quad (4.6)$$

Therefore, using Def. 9,

$$[(b \preceq_1 a) \wedge (c \preceq_2 b) \wedge (d \preceq_1 b) \wedge (c \preceq_2 d) \wedge (e \preceq_2 c)], \quad (4.7)$$

$$\implies [(d \preceq_1 a) \wedge (c \preceq_2 d)] \implies (a \tilde{\preceq}_1 d \tilde{\preceq}_2 c) \quad (4.8)$$

and

$$[(b \preceq_1 a) \wedge (e \preceq_2 b)] \implies (a \tilde{\preceq}_1 b \tilde{\preceq}_2 e). \quad (4.9)$$

Hence, \diamond is transitive and $(P, \tilde{\tilde{\diamond}})$ is also a binary poset.

Definition 9. Let (P, \diamond) be a binary poset. Then, $(P, \tilde{\tilde{\diamond}})$ is known as the dual of (P, \diamond) .

Definition 10 (Self dual of a binary poset). If (P, \diamond) is a binary poset and $(P, \diamond) \cong (P, \tilde{\tilde{\diamond}})$, then (P, \diamond) is known as self dual poset.

Theorem 10. *Let X be a non-empty set. Then, $(P(X), \diamond)$ is a self dual poset, where $\diamond = (\subseteq, \supseteq)$.*

Proof: Define the mapping $\psi : P(X) \rightarrow P(\tilde{X})$ by $\psi(A) = X - A, \forall A \in P(X)$. Let $A, B \in P(X)$ such that $A = B$ where $P(\tilde{X}) = (P(X), \tilde{\diamond})$. Therefore,

$$X - A = X - B \implies \psi(A) = \psi(B). \quad (4.10)$$

Hence ψ is well defined and injective. Now, let $A \in P(\tilde{X})$. Therefore

$$X - A \in P(X) \implies \psi(X - A) = X - (X - A) = A. \quad (4.11)$$

Hence ψ is surjective. Let $A, B, C \in P(X)$ such that $A \subseteq B \subseteq C$. Therefore,

$$\begin{aligned} A \subseteq B \subseteq C &\implies X - B \subseteq X - A \subseteq C \\ &\implies \psi(B) \subseteq \psi(A) \subseteq C \\ &\implies C \supseteq \psi(A) \supseteq \psi(B). \end{aligned} \quad (4.12)$$

Also,

$$\begin{aligned} A \subseteq B \subseteq C &\implies A \subseteq X - C \subseteq X - B \\ &\implies A \subseteq \psi(C) \subseteq \psi(B) \\ &\implies \psi(B) \supseteq \psi(C) \supseteq A. \end{aligned} \quad (4.13)$$

Therefore

$$C \supseteq \psi(A) \supseteq \psi(B) \text{ and } \psi(B) \supseteq \psi(C) \supseteq A \implies \psi(A) \supseteq \psi(B) \supseteq \psi(C). \quad (4.14)$$

Hence, $\psi : P(X) \rightarrow P(\tilde{X})$ is an isomorphism and $P(X) \cong P(\tilde{X})$. Therefore, $(P(X), \diamond)$ is self dual.

Theorem 11. *Let (P, \diamond) be a binary poset and let $(P, \tilde{\diamond})$ be the dual of (P, \diamond) . Then, (P, \diamond) is isomorphic to $(P, \tilde{\diamond})$, i.e. $(P, \diamond) \cong (P, \tilde{\diamond})$.*

Proof: Define $\psi : (P, \diamond) \rightarrow (P, \tilde{\diamond})$ by $\psi(a) = a, \forall a \in P$. Clearly, $\psi(a)$ is well defined and bijective. Let $\diamond = (\preceq_1, \preceq_2), \tilde{\diamond} = (\tilde{\preceq}_1, \tilde{\preceq}_2), \tilde{\tilde{\diamond}} = (\tilde{\tilde{\preceq}}_1, \tilde{\tilde{\preceq}}_2)$ (by Def. 9). Now, Let $a, b, c \in X$ such that

$$\begin{aligned} c \preceq_2 b \preceq_1 a &\iff a \tilde{\preceq}_1 b \tilde{\preceq}_2 c \\ &\iff c \tilde{\preceq}_2 b \tilde{\preceq}_1 a \\ &\iff \psi(c) \tilde{\preceq}_2 \psi(b) \tilde{\preceq}_1 \psi(a). \end{aligned} \quad (4.15)$$

Therefore, $\psi(P, \diamond) \rightarrow (P, \tilde{\diamond})$ is an isomorphism and thus $(P, \diamond) \cong (P, \tilde{\diamond})$.

5. GALOIS CONNECTIONS

Let (P, \diamond) be a binary poset where $\diamond = (\preceq_1, \preceq_2)$ and $a, b \in P$. Then, define the notation $a \diamond b = a \preceq_1 b, a \preceq_2 b$. Throughout the sequel we let $\diamond = (\preceq_1, \preceq_2), \tilde{\diamond} = (\preceq_3, \preceq_4)$ and $\tilde{\tilde{\diamond}} = (\preceq_3, \preceq_4)$.

Definition 11 (Galois connection). *Let (P, \diamond) and $(Q, \tilde{\diamond})$ be two binary posets. Let $\psi^* : (P, \diamond) \rightarrow (Q, \tilde{\diamond})$ and $\psi_* : (Q, \tilde{\diamond}) \rightarrow (P, \diamond)$ be a pair of functions. For $a \in P$ and $b \in Q$,*

if

$$\psi^*(a) \tilde{\diamond} b \iff a \diamond \psi_*(b), \quad (5.1)$$

then the pair ψ^*, ψ_* forms a Galois connection between binary posets (P, \diamond) and $(Q, \tilde{\diamond})$ and is denoted by $\text{Gal}(\psi^*, \psi_*)$ or more generally, $\text{Gal}(\psi^*, \psi_*) : (P, \diamond) \rightarrow (Q, \tilde{\diamond})$.

Following are some examples of a pair of binary posets and functions that form a Galois connection.

- (1) Let the map ψ be an binary poset isomorphism between (P, \diamond) and $(Q, \tilde{\diamond})$. Then, $\text{Gal}(\psi, \psi^{-1}) : (P, \diamond) \rightarrow (Q, \tilde{\diamond})$ is a Galois connection.
- (2) Let (\mathbb{N}, \diamond) and $(\mathbb{Q}^+, \tilde{\diamond})$ be to binary posets where $\diamond = (\preceq, \preceq)$. Let $\psi^* : \mathbb{N} \rightarrow \mathbb{Q}^+$ be that standard embedding of the natural numbers into the rationals and $\psi_* : \mathbb{Q}^+ \rightarrow \mathbb{N}$ be the map a positive rationals to the natural numbers corresponding to its integral part. Then, $\text{Gal}(\psi^*, \psi_*) : (\mathbb{N}, \diamond) \rightarrow (\mathbb{Q}^+, \tilde{\diamond})$ is a Galois connection.
- (3) Let (P, \diamond) be an arbitrary binary poset and let $(\{0\}, \tilde{\diamond})$ be a singleton binary poset where $\tilde{\diamond} = (=, =)$. Let $\psi^* : (P, \diamond) \rightarrow (\{0\}, \tilde{\diamond})$ be a trivial function mapping all elements of P to $\{0\}$ and $\psi_* : (\{0\}, \tilde{\diamond}) \rightarrow (P, \diamond)$ be another dunction mapping 0 to particular elements of P . Then, $\text{Gal}(\psi^*, \psi_*) : (P, \diamond) \rightarrow (\{0\}, \tilde{\diamond})$ is a Galois connection.

Following is one of the alternative definition of Def. 12.

Theorem 12. Let (P, \diamond) and $(Q, \tilde{\diamond})$ be two binary posets. Let $\psi^* : (P, \diamond) \rightarrow (Q, \tilde{\diamond})$ and $\psi_* : (Q, \tilde{\diamond}) \rightarrow (P, \diamond)$ be a pair of functions. Then, $\text{Gal}(\psi^*, \psi_*) : (P, \diamond) \rightarrow (Q, \tilde{\diamond})$ is Galois connection if:-

- (1) ψ^* and ψ_* both are isotones, and
- (2) $\forall a \in P$ and $b \in Q$, $a \diamond \psi^*(\psi_*(a))$ and $\psi_*(\psi^*(b)) \tilde{\diamond} b$.

Proof: Suppose that $\text{Gal}(\psi^*, \psi_*) : (P, \diamond) \rightarrow (Q, \tilde{\diamond})$ is Galois connection. Then, by definition we have

$$\psi^*(a) \tilde{\diamond} \psi^*(a) \iff a \diamond \psi_* \circ \psi^*(a). \quad (5.2)$$

Since $\tilde{\diamond}$ is reflexive, $\psi^*(a) \tilde{\diamond} \psi^*(a)$ holds true and thus $a \diamond \psi_* \circ \psi^*(a)$. By a similar argument, we get $\psi_* \circ \psi^*(b) \tilde{\diamond} b$. Now, let $a' \in P$ and $a \diamond a'$. Since we have just shown that $a' \diamond \psi_* \circ \psi^*(a')$, we get $a \diamond \psi_* \circ \psi^*(a')$. But, by Def. 12 we have

$$\psi^*(a) \tilde{\diamond} \psi^*(a') \iff a \diamond \psi_* \circ \psi^*(a'), \quad (5.3)$$

therefore, $\psi^* \tilde{\diamond} \psi^*(a,)$ and thus ψ^* is a isotone. By a similar argument, we can show that ψ_* is also a isotone.

So far, we are dealing with Galois connections that are equipped with binary partially ordered relations that are not same. When $\diamond = \tilde{\diamond}$, we can define monotone and antitone Galois connection as follows.

Definition 12 (Monotone Galois connection). Let (P, \diamond) and (Q, \diamond) be two binary posets. Let $\psi^* : (P, \diamond) \rightarrow (Q, \diamond)$ and $\psi_* : (Q, \diamond) \rightarrow (P, \diamond)$ be a pair of functions. For $a \in P$ and $b \in Q$, if

$$\psi^*(a) \diamond b \iff a \diamond \psi_*(b), \quad (5.4)$$

then the pair ψ^*, ψ_* forms a monotone Galois connection between binary posets (P, \diamond) and (Q, \diamond) and is denoted by $\text{Gal}_{\text{mon}}(\psi^*, \psi_*)$ or more generally, $\text{Gal}_{\text{mon}}(\psi^*, \psi_*) : (P, \diamond) \rightarrow (Q, \diamond)$.

Definition 13 (Antitone Galois connection). Let (P, \diamond) and (Q, \diamond) be two binary posets. Let $\psi^* : (P, \diamond) \rightarrow (Q, \diamond)$ and $\psi_* : (Q, \diamond) \rightarrow (P, \diamond)$ be a pair of functions. For $a \in P$

and $b \in Q$, if

$$b \diamond \psi^*(a) \iff a \diamond \psi_*(b). \quad (5.5)$$

Then the pair ψ^*, ψ_* forms a antitone Galois connection between binary posets (P, \diamond) and (Q, \diamond) and is denoted by $\text{Gal}_{\text{ant}}(\psi^*, \psi_*)$ or more generally,

$$\text{Gal}_{\text{ant}}(\psi^*, \psi_*) : (P, \diamond) \rightarrow (Q, \tilde{\diamond}). \quad (5.6)$$

Proposition 1. Let (P, \diamond) , $(Q, \tilde{\diamond})$ and $(R, \tilde{\tilde{\diamond}})$ be any three binary posets. Then,

$$\begin{aligned} \text{Gal}(\psi^*, \psi_*) : (P, \diamond) \rightarrow (Q, \tilde{\diamond}) \wedge \text{Gal}(\phi^*, \phi_*) : (Q, \tilde{\diamond}) \rightarrow (R, \tilde{\tilde{\diamond}}) \\ \implies \text{Gal}(\phi^* \circ \psi^*, \phi_* \circ \psi_*) : (P, \diamond) \rightarrow (R, \tilde{\tilde{\diamond}}). \end{aligned} \quad (5.7)$$

Proposition 2. We have

$$\text{Gal}(\psi^*, \psi^1) : (P, \diamond) \rightarrow (Q, \tilde{\diamond}) \wedge \text{Gal}(\psi^*, \psi^2) : (P, \diamond) \rightarrow (Q, \tilde{\diamond}) \implies \psi^1 = \psi^2 \quad (5.8)$$

and

$$\text{Gal}(\psi_1, \psi_*) : (P, \diamond) \rightarrow (Q, \tilde{\diamond}) \wedge \text{Gal}(\psi_2, \psi_*) : (P, \diamond) \rightarrow (Q, \tilde{\diamond}) \implies \psi_1 = \psi_2. \quad (5.9)$$

Proposition 3. Galois connections are not necessarily symmetric. Thus,

$$\text{Gal}(\psi^*, \psi_*) : (P, \diamond) \rightarrow (Q, \tilde{\diamond}) \not\equiv \text{Gal}(\psi_*, \psi^*) : (Q, \tilde{\diamond}) \rightarrow (P, \diamond). \quad (5.10)$$

Proof of above propositions follow trivially from Def. 12 [8].

6. CONCLUSIONS AND/OR DISCUSSIONS

In this paper, we have derived some results that would help us to study partially ordered sets under two partially ordered relations simultaneously. We call this the binary poset. Furthermore, it has been show that binary posets exhibit same properties as posets even under come complicated properties such as the uniqueness of maximal and minimal element, existence of isotone and isotone inverse. Some other properties such as isomorphism, duality principle and Galois connections for binary posets are also derived. The derived results are applied to obtain some results on the power set. The next step in research in this direction would be to construct the concept of binary chains and most importantly, binary lattices [3],[4]. The Hasse diagrams will be different too. It has been known widely in the literature that Galois connections can be used to study posets under the category theory [5-8] and many other structures [9-12]. The structures that we can derived in category theory from the Galois connections of binary posets can be mathematically interesting to study. Work in this direction is under progress.

7. ACKNOWLEDGEMENTS

The author would like to thank anonymous referees for their kind and valuable remarks throughout the publication of this paper.

REFERENCES

- [1] Miraglia, F., An introduction to partially ordered structures and sheaves. (2006) Polimetrica sas.
- [2] Smith, P., The Galois connection between syntax and semantics, (2006), University of Cambridge.
- [3] K. Engel, Sperner Theory, Cambridge University Press (1997).
- [4] R. Stanley, Algebraic Combinatorics: Walks, Trees, Tableaux, and More, Springer (2013)
- [5] Gähler, W., 2004. Galois Connections in Category Theory, Topology and Logic. In Galois Connections and Applications (pp. 425-452). Springer, Dordrecht.

- [6] A.V.S.N. Murty, "Structures on Galois connections", Italian journal of pure and applied mathematics – N. 44–2020 (150–154).
- [7] Melton, A., Schröder, B.S. and Strecker, G.E., Lagois connections—a counterpart to Galois connections. Theoretical Computer Science, (1994) 136(1), pp.79-107.
- [8] Shmuely, Z., The structure of Galois connections. Pacific Journal of Mathematics, (1974) 54(2), pp.209-225.
- [9] Bělohlávek, R., Fuzzy galois connections. Mathematical Logic Quarterly, (1999) 45(4), pp.497-504.
- [10] Bělohlávek, R., Lattices of fixed points of fuzzy Galois connections. Mathematical Logic Quarterly: Mathematical Logic Quarterly,(2001) 47(1), pp.111-116.
- [11] Száz, A., A particular Galois connection between relations and set functions. Acta Univ. Sapientiae, Math, 6, pp.73-91.
- [12] Yao, W. and Lu, L.X., 2009. Fuzzy Galois connections on fuzzy posets. Mathematical Logic Quarterly, (2014) 55(1), pp.105-112.

OMPRAKASH ATAŁE

SCHOOL OF MATHEMATICAL SCIENCES, MOOLJI JAITHA COLLEGE JALGAON-425001, INDIA

Email address: atale.om@outlook.com