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A GENERALIZATION OF $(\in, \in \vee q)$-FUZZY IDEALS IN TERNARY SEMIGROUPS

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ABSTRACT. The notion of $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy left (right, lateral) ideal is introduced, and related properties are investigated. Also, characterizations of an $(\in, \in \vee\, q_0^\delta)$-fuzzy left (resp., right, lateral) ideal are provided. Moreover, relations between $(\in, \in \vee\, q_0^\delta)$-fuzzy left (resp., right, lateral) ideal and $(q_0^\delta, \in \vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal are established.

1. INTRODUCTION

The literature of ternary algebraic system was introduced by Lehmer [6] in 1932. He investigated certain ternary algebraic systems called triplexex which turn out to be ternary groups. The notion of ternary semigroups was also known to Banach (cf. [7]) who is credited with an example of a ternary semigroup which does not reduce to a semigroup. Los [7] showed that every ternary semigroup can be imbedded in a semigroup. Sioson [16] studied ternary semigroups with special reference to ideals and radicals. An extensive amount of applications of fuzzy set theory introduced by Zadeh (see [17]) have been found in diverse fields such as expert system, computer science, artificial intelligence, control engineering, information sciences, operation research, coding theory, robotics etc. After then, reconsideration of the concepts of classical mathematics began. On the other hand, because of the importance of group theory in mathematics as well as its applications in many areas, the notion of fuzzy subgroup was introduced by Rosenfeld [14]. Since then many researchers have been engaged to review various concepts and results from the realm of abstract algebra in broader framework of fuzzy setting (for example see [15]). The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [11], played a vital role to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bhakat and Das [1]. In particular, $(\in, \in \vee\, q)$ -fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. Also, Muhiuddin et al. and Jun et al. have applied the fuzzy set theory and related notions to the different aspects in semigroups (see for e.g., [2, 3, 4, 8, 9, 10])

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In ternary semigroups, the concept of (α, β)-fuzzy ternary subsemigroups, which is studied in the paper [12], is also important and useful generalization of the well-known concepts, called fuzzy ternary subsemigroups. Furthermore, Kar and Sarkar [5] introduced the concept of fuzzy ideals of ternary semigroups.

In this paper, we deal with more general form of (α, β)-fuzzy left (resp., right, lateral) ideals in [12, 13]. We introduce the notions of $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy left (resp., right, lateral) ideals in ternary semigroups, and investigate related properties. We provide characterizations of $(\in, \in \vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideals. We provide a condition for an $(\in, \in \vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal to be an $(\in, \in)$-fuzzy left (resp., right, lateral) ideal. We establish relations between $(\in, \in \vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal and $(q_0^\delta, \in \vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal.

2. Preliminaries

A non-empty subset A of a ternary semigroup X is called

- a ternary subsemigroup of X if $AAA \subseteq A$, that is, $abc \in A$ for all $a, b, c \in A$,
- a left (resp., right, lateral) ideal of X if $SSA \subseteq A$ (resp., $ASS \subseteq A$, $SAS \subseteq A$),
- a two-sided ideal of X if it is both left and right ideal of X,
- an ideal of X if it is a left, right and lateral ideal of X.

For two fuzzy set μ and ν in X, we say $\mu \leq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in X$. We define $\mu \wedge \nu$ and $\mu \vee \nu$ as follows:

$$\mu \wedge \nu : X \to [0, 1],\ x \mapsto \min\{\mu(x), \nu(x)\}$$

and

$$\mu \vee \nu : X \to [0, 1],\ x \mapsto \max\{\mu(x), \nu(x)\}$$

respectively.

A fuzzy set μ in a set X of the form

$$\mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{2.1}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, we say that

- $x_t \in \mu$ (resp., $x_t\, q\, \mu$) (see [11]) if $\mu(x) \geq t$ (resp., $\mu(x) + t > 1$). In this case, x_t is said to *belong to* (resp., *be quasi-coincident with*) a fuzzy set μ.
- $x_t \in \vee\, q\, \mu$ (resp., $x_t \in \wedge\, q\, \mu$) (see [11]) if $x_t \in \mu$ or $x_t\, q\, \mu$ (resp., $x_t \in \mu$ and $x_t\, q\, \mu$).

Let $\delta \in (0, 1]$. For a fuzzy point x_t and a fuzzy set μ in a set X, we say that

- x_t is a δ-*quasi-coincident* with μ, written $x_t\, q_0^\delta\, \mu$, if $\mu(x) + t > \delta$,
- $x_t \in \vee q_0^\delta\, \mu$ (resp., $x_t \in \wedge\, q_0^\delta \mu$) if $x_t \in \mu$ or $x_t\, q_0^\delta\, \mu$ (resp., $x_t \in \mu$ and $x_t\, q_0^\delta\, \mu$).

Obviously, $x_t\, q\, \mu$ implies $x_t\, q_0^\delta\, \mu$. If $\delta = 1$, then the δ-quasi-coincident with μ is the quasi-coincident with μ, that is, $x_t\, q_0^1\, \mu = x_t\, q\, \mu$.

For $\alpha \in \{\in, q, \in \vee\, q, \in \wedge\, q, \in \vee q_0^\delta, \in \wedge\, q_0^\delta\}$, we say that $x_t\, \overline{\alpha}\, \mu$ if $x_t\, \alpha\, \mu$ does not hold.

3. Generalized fuzzy ideals in ternary semigroups

In what follows, let δ be an element of $(0, 1]$ and let X be a semigroup and $\widetilde{\alpha}$ and $\widetilde{\beta}$ denote any one of $\in$, q_0^δ, $\in \vee q_0^\delta$ and $\in \wedge\, q_0^\delta$ unless otherwise specified.

Definition 3.1. A fuzzy set μ in X is called an $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy left (resp., right, lateral) ideal of X, where $\widetilde{\alpha} \neq \in\wedge\, q_0^\delta$, if for all $x, y, z \in X$ and $t \in (0, \delta]$,

$$z_t \,\widetilde{\alpha}\, \mu \;\Rightarrow\; (xyz)_t \,\widetilde{\beta}\, \mu \text{ (resp., } (zxy)_t \,\widetilde{\beta}\, \mu,\ (xzy)_t \,\widetilde{\beta}\, \mu). \tag{3.1}$$

If μ is both an $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy left ideal and an $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy right ideal of X, we say that μ is an $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy two-sided ideal of X. If μ is an $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy left ideal, an $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy right ideal and an $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy lateral ideal of X, we say that μ is an $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy ideal of X.

Let μ be a fuzzy set in X such that $\mu(x) \leq \frac{\delta}{2}$ for all $x \in X$. Let $x \in X$ and $t \in (0, \delta]$ be such that $x_t \in\wedge\, q_0^\delta\, \mu$. Then $\mu(x) \geq t$ and $\mu(x) + t > \delta$. It follows that $\delta < \mu(x) + t \leq 2\mu(x)$, so that $\mu(x) \geq \frac{\delta}{2}$. This means that $\{x_t \mid x_t \in\wedge\, q_0^\delta\, \mu\} = \emptyset$. Hence the case $\widetilde{\alpha} = \in\wedge\, q_0^\delta$ should be omitted.

If a fuzzy set μ in X satisfies the condition (3.1) for all $x, y, z \in X$ and $t \in (0, \frac{\delta}{2}]$, then we say that μ is a $\frac{\delta}{2}$-lower $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy left (resp., right, lateral) ideal of X. If a fuzzy set μ in X satisfies the condition (3.1) for all $x, y, z \in X$ and $t \in (\frac{\delta}{2}, 1]$, then we say that μ is a $\frac{\delta}{2}$-upper $(\widetilde{\alpha}, \widetilde{\beta})$-fuzzy left (resp., right, lateral) ideal of X.

Example 3.2. Let $X = \{a, b, c, d, e\}$ be a ternary semigroup in which $xyz = (x * y) * z = x * (y * z)$ for all $x, y, z \in X$ and $*$ is defined by the following Cayley table:

$*$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

Define a fuzzy set μ in X as follows:

$$\mu : X \to [0, 1],\ x \mapsto \begin{cases} 0.40 & \text{if } x = a, \\ 0.65 & \text{if } x = b, \\ 0.25 & \text{if } x = c, \\ 0.45 & \text{if } x = d, \\ 0.55 & \text{if } x = e. \end{cases} \tag{3.2}$$

Then μ is an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal of X for $\delta = 0.8$. Note that $e_{0.35} \in \mu$, $(ecc)_{0.35}\, \overline{\in\vee q_0^\delta}\, \mu$ and $(cec)_{0.35}\, \overline{\in\vee q_0^\delta}\, \mu$ for $\delta = 0.8$. Thus μ is neither an $(\in, \in\vee q_0^\delta)$-fuzzy right ideal of X nor an $(\in, \in\vee q_0^\delta)$-fuzzy lateral ideal of X for $\delta = 0.8$.

Note that every $(\in, \in)$-fuzzy ideal of X is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of X. But the converse is not true. In fact, the $(\in, \in\vee q_0^\delta)$-fuzzy left ideal μ of X for $\delta = 0.8$ in Example 3.2 is not an $(\in, \in)$-fuzzy left ideal of X for $\delta = 0.8$ since $b_{0.6} \in \mu$ but $(dcb)_{0.6}\, \overline{\in}\, \mu$. Obviously, if $\delta_1 \leq \delta_2$ in $(0, 1]$ then every $(\in, \in\vee q_0^{\delta_2})$-fuzzy left ideal of X is an $(\in, \in\vee q_0^{\delta_1})$-fuzzy left ideal of X. But the converse is not true. In fact, the $(\in, \in\vee q_0^\delta)$-fuzzy left ideal μ of X for $\delta = 0.8$ in Example 3.2 is not an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal of X for $\delta = 0.98$.

Example 3.3. Let $X = \{0, a, b, c, 1\}$ be a ternary semigroup in which $xyz = (x * y) * z = x * (y * z)$ for all $x, y, z \in X$ and $*$ is defined by the following Cayley table:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	b	b	b
c	0	0	b	c	c
1	0	a	b	c	1

Define a fuzzy set μ in X as follows:

$$\mu : X \to [0,1],\ x \mapsto \begin{cases} 0.77 & \text{if } x = 0, \\ 0.22 & \text{if } x = a, \\ 0.35 & \text{if } x = b, \\ 0.44 & \text{if } x = c, \\ 0.22 & \text{if } x = 1. \end{cases} \tag{3.3}$$

Then μ is an $(\in, \in\vee q_0^\delta)$-fuzzy right ideal of X for $\delta = 0.7$. But μ is neither an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal of X nor an $(\in, \in\vee q_0^\delta)$-fuzzy lateral ideal of X for $\delta = 0.7$ since $c_{0.13} \in \mu$, $(1ac)_{0.13}\, \overline{\in\vee q_0^\delta}\mu$ and $(ac1)_{0.13}\, \overline{\in\vee q_0^\delta}\mu$ for $\delta = 0.7$. Moreover we know that

(i) μ is not an $(\in, \in)$-fuzzy right ideal of X since $c_{0.4} \in \mu$ and $(c1b)_{0.4}\, \overline{\in}\, \mu$.
(ii) μ is not an $(\in, q_0^\delta)$-fuzzy right ideal of X since $c_{0.3} \in \mu$ and $(c1b)_{0.3}\, \overline{q_0^\delta}\, \mu$ for $\delta = 0.7$.
(iii) μ is not a (q_0^δ, q_0^δ)-fuzzy right ideal of X since $c_{0.33} q_0^\delta \mu$ and $(cbb)_{0.33}\, \overline{q_0^\delta}\, \mu$ for $\delta = 0.7$.
(iv) μ is not a $(q_0^\delta, \in)$-fuzzy right ideal of X since $a_{0.8} q_0^\delta \mu$ and $(a00)_{0.8}\, \overline{\in}\, \mu$ for $\delta = 0.7$.
(v) μ is not an $(\in, \in\wedge q_0^\delta)$-fuzzy right ideal of X since $c_{0.38} \in \mu$ and $(c1b)_{0.38}\, \overline{\in\wedge q_0^\delta}\, \mu$ for $\delta = 0.7$.
(vi) μ is not a $(q_0^\delta, \in\wedge q_0^\delta)$-fuzzy right ideal of X since $c_{0.31} q_0^\delta \mu$ and $(cbb)_{0.31}\, \overline{\in\wedge q_0^\delta}\, \mu$ for $\delta = 0.7$.
(vii) μ is not an $(\in\vee q_0^\delta, \in)$-fuzzy right ideal of X since $c_{0.39} \in\vee q_0^\delta \mu$ and $(c1b)_{0.39}\, \overline{\in}\, \mu$ for $\delta = 0.7$.
(viii) μ is not an $(\in\vee q_0^\delta, q_0^\delta)$-fuzzy right ideal of X since $c_{0.32} \in\vee q_0^\delta \mu$ and $(cbb)_{0.32}\, \overline{q_0^\delta}\, \mu$ for $\delta = 0.7$.
(ix) μ is not an $(\in\vee q_0^\delta, \in\wedge q_0^\delta)$-fuzzy right ideal of X since $c_{0.34} \in\vee q_0^\delta \mu$ and $(cbb)_{0.34}\, \overline{\in\wedge q_0^\delta}\, \mu$ for $\delta = 0.7$.

Theorem 3.1. *For a fuzzy set μ in X, the following are equivalent.*

(i) *μ is an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal of X.*
(ii) $(\forall x, y, z \in X)\left(\mu(xyz) \geq \min\{\mu(z), \frac{\delta}{2}\}\right)$.

Proof. Let μ be an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal of X. Suppose there exist $a, b, c \in X$ such that $\mu(abc) < \min\{\mu(c), \frac{\delta}{2}\}$. Take $t \in (0, \delta]$ such that $\mu(abc) < t \leq \min\{\mu(c), \frac{\delta}{2}\}$. Then $c_t \in \mu$ but $(abc)_t\, \overline{\in}\, \mu$ and $\mu(abc) + t \leq \delta$, i.e., $(abc)_t\, \overline{q_0^\delta}\, \mu$. This is a contradiction, and therefore (ii) is valid.

Conversely, assume that $\mu(xyz) \geq \min\{\mu(z), \frac{\delta}{2}\}$ for all $x, y, z \in X$. Let $z_t \in \mu$ for $t \in (0, \delta]$. Then $\mu(z) \geq t$, and so $\mu(xyz) \geq \min\{\mu(z), \frac{\delta}{2}\} \geq \min\{t, \frac{\delta}{2}\}$. If $t \leq \frac{\delta}{2}$, then $\mu(xyz) \geq t$, i.e., $(xyz)_t \in \mu$. If $t > \frac{\delta}{2}$, then $\mu(xyz) \geq \frac{\delta}{2}$ and so $\mu(xyz) + t > \frac{\delta}{2} + \frac{\delta}{2} = \delta$, i.e., $(xyz)_t\, q_0^\delta\, \mu$. Hence $(xyz)_t \in\vee q_0^\delta \mu$, and therefore μ is an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal of X. $\square$

Similarly, we have the following theorems.

Theorem 3.2. *For a fuzzy set μ in X, the following are equivalent.*

(i) *μ is an $(\in, \in\vee q_0^\delta)$-fuzzy right ideal of X.*
(ii) $(\forall x, y, z \in X)\left(\mu(xyz) \geq \min\{\mu(x), \frac{\delta}{2}\}\right)$.

Theorem 3.3. *For a fuzzy set μ in X, the following are equivalent.*

(i) *μ is an $(\in, \in\vee q_0^\delta)$-fuzzy lateral ideal of X.*
(ii) $(\forall x, y, z \in X)\left(\mu(xyz) \geq \min\{\mu(y), \frac{\delta}{2}\}\right)$.

Corollary 3.4. *For a fuzzy set μ in X, the following are equivalent.*

(i) *μ is an $(\in, \in\vee q_0^\delta)$-fuzzy two-sided ideal of X.*
(ii) $(\forall x, y, z \in X)\left(\mu(xyz) \geq \min\{\mu(x), \frac{\delta}{2}\},\ \mu(xyz) \geq \min\{\mu(z), \frac{\delta}{2}\}\right)$.

Theorem 3.5. *A fuzzy set μ in X is an $(\in, \in\vee q_0^\delta)$-fuzzy lateral ideal of X if and only if the set*

$$U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$$

is a lateral ideal of X for all $t \in (0, \frac{\delta}{2}]$.

Proof. Assume that μ is an $(\in, \in\vee q_0^\delta)$-fuzzy lateral ideal of X. Let $t \in (0, \frac{\delta}{2}]$, $x, z \in X$ and $y \in U(\mu; t)$. Then $\mu(y) \geq t$, and so

$$\mu(xyz) \geq \min\{\mu(y), \tfrac{\delta}{2}\} \geq \min\{t, \tfrac{\delta}{2}\} = t.$$

Hence $xyz \in U(\mu; t)$, and therefore $U(\mu; t)$ is a lateral ideal of X.

Conversely, let μ be a fuzzy set in X such that $U(\mu; t)$ is a lateral ideal of X for all $t \in (0, \frac{\delta}{2}]$. Suppose that there are elements a, b and c of X such that

$$\mu(abc) < \min\{\mu(b), \tfrac{\delta}{2}\},$$

and take $t \in (0, \delta]$ such that $\mu(abc) < t \leq \min\{\mu(b), \frac{\delta}{2}\}$. Then $b \in U(\mu; t)$ and $t \leq \frac{\delta}{2}$, which implies that $abc \in U(\mu; t)$ since $U(\mu; t)$ is a lateral ideal of X. This induces $\mu(abc) \geq t$, and this is a contradiction. Hence $\mu(xyz) \geq \min\{\mu(y), \frac{\delta}{2}\}$ for all $x, y, z \in X$, and therefore μ is an $(\in, \in\vee q_0^\delta)$-fuzzy lateral ideal of X by Theorem 3.3. □

Similarly, we have the following theorem.

Theorem 3.6. *A fuzzy set μ in X is an $(\in, \in\vee q_0^\delta)$-fuzzy left (resp., right) ideal of X if and only if the set*

$$U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$$

is a left (resp., right) ideal of X for all $t \in (0, \frac{\delta}{2}]$.

Corollary 3.7 ([12])**.** *A fuzzy set μ in X is an $(\in, \in\vee q)$-fuzzy left (resp., right, lateral) ideal of X if and only if the set*

$$U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}$$

is a left (resp., right, lateral) ideal of X for all $t \in (0, 0.5]$.

Theorem 3.8. *If μ is a nonzero $(q_0^\delta, \in)$-fuzzy lateral ideal of X, then the set*

$$X_0 := \{x \in X \mid \mu(x) > 0\}$$

is a lateral ideal of X.

Proof. Let $x, z \in X$ and $y \in X_0$. Then $\mu(y) > 0$, and so $\mu(y) + \delta > \delta$, that is, $y_\delta\, q_0^\delta\, \mu$. It follows that $(xyz)_\delta \in \mu$, i.e., $\mu(xyz) \geq \delta > 0$. Thus $xyz \in X_0$. This completes the proof. □

Similarly, we have the following theorem.

Theorem 3.9. *If μ is a nonzero $(q_0^\delta, \in)$-fuzzy left (resp., right) ideal of X, then the set*

$$X_0 := \{x \in X \mid \mu(x) > 0\}$$

is a left (resp., right) ideal of X.

Theorem 3.10. *If μ is a nonzero (q_0^δ, q_0^δ)-fuzzy lateral ideal of X, then the set*

$$X_0 := \{x \in X \mid \mu(x) > 0\}$$

is a lateral ideal of X.

Proof. Let $x, z \in X$ and $y \in X_0$. Then $\mu(y) > 0$, and hence $\mu(y) + \delta > \delta$, that is, $y_\delta\, q_0^\delta\, \mu$. It follows that $(xyz)_\delta\, q_0^\delta\, \mu$, i.e., $\mu(xyz) + \delta > \delta$. Thus $\mu(xyz) > 0$, and so $xyz \in X_0$. This completes the proof. □

Similarly, we have the following theorem.

Theorem 3.11. *If μ is a nonzero (q_0^δ, q_0^δ)-fuzzy left (resp., right) ideal of X, then the set*

$$X_0 := \{x \in X \mid \mu(x) > 0\}$$

is a left (resp., right) ideal of X.

Theorem 3.12. *If μ is a nonzero $(\in, \in \vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal of X, then the set*

$$X_0 := \{x \in X \mid \mu(x) > 0\}$$

is a left (resp., right, lateral) ideal of X.

We investigate relations between $(\in, \in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideals and $(q_0^\delta, \in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideals.

Theorem 3.13. *Every $(q_0^\delta, \in \vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal of X is an $(\in, \in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal of X.*

Proof. Let μ be a $(q_0^\delta, \in \vee q_0^\delta)$-fuzzy left ideal of X and let $x, y, z \in X$ and $t \in (0, \delta]$ such that $z_t \in \mu$. Then $\mu(z) \geq t$. Assume that $(xyz)_t\, \overline{\in\vee q_0^\delta}\, \mu$. Then $\mu(xyz) < t$ and $\mu(xyz) + t \leq \delta$, which imply that $\mu(xyz) < \frac{\delta}{2}$. Hence $\mu(xyz) < \min\{t, \frac{\delta}{2}\}$, and so

$$\delta - \mu(xyz) > \delta - \min\{t, \tfrac{\delta}{2}\} = \max\{\delta - t, \tfrac{\delta}{2}\} \geq \max\{\delta - \mu(z), \tfrac{\delta}{2}\}. \tag{3.4}$$

Thus there exists $r \in (0, \delta]$ such that

$$\max\{\delta - \mu(z), \tfrac{\delta}{2}\} < r \leq \delta - \mu(xyz). \tag{3.5}$$

The left inequality in (3.5) induces $z_r\, q_0^\delta\, \mu$. Since μ is a $(q_0^\delta, \in\vee q_0^\delta)$-fuzzy left ideal of X, we have $(xyz)_r \in\vee q_0^\delta\, \mu$. On the other hand, we get $\mu(xyz) + r \leq \delta$, i.e., $(xyz)_r\, \overline{q_0^\delta}\, \mu$, and $\mu(xyz) \leq \delta - r < \delta - \frac{\delta}{2} = \frac{\delta}{2} < \delta$, i.e., $(xyz)_r\, \overline{\in}\, \mu$ from the right inequality in (3.5). Hence $(xyz)_r\, \overline{\in\vee q_0^\delta}\, \mu$, and it is a contradiction. Therefore $(xyz)_r \in\vee q_0^\delta \mu$ and thus μ is an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal of X. The cases of right and lateral can be proved by the similar way. □

Corollary 3.14 ([13])**.** *Every $(q, \in\vee q)$-fuzzy left (resp., right, lateral) ideal of X is an $(\in, \in\vee q)$-fuzzy left (resp., right, lateral) ideal of X.*

Theorem 3.15. *If every fuzzy point has the value t in $(0, \frac{\delta}{2}]$, then every $(\in, \in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal of X is a $(q_0^\delta, \in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal of X.*

Proof. Let μ be an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal of X. let $x, y, z \in X$ and $t \in (0, \frac{\delta}{2}]$ be such that $z_t\, q_0^\delta\, \mu$. Then $\mu(z) + t > \delta$, and so $\mu(z) > \delta - t \geq t$, i.e., $z_t \in \mu$. Since μ is an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal of X, it follows that $(xyz)_t \in\vee q_0^\delta \mu$. Hence μ is a $(q_0^\delta, \in\vee q_0^\delta)$-fuzzy left ideal of X. The cases of right and lateral can be proved by the similar way. □

Corollary 3.16 ([13]). *If every fuzzy point has the value t in $(0, 0.5]$, then every $(\in, \in\vee q)$-fuzzy left (resp., right, lateral) ideal of X is a $(q, \in\vee q)$-fuzzy left (resp., right, lateral) ideal of X.*

Theorem 3.17. *For a subset L of X and $\varepsilon \geq \frac{\delta}{2}$, consider a fuzzy set μ_ε in X as follows:*

$$\mu_\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} \varepsilon & \text{if } x \in L, \\ 0 & \text{if } x \notin L. \end{cases}$$

If L is a left (resp., right, lateral) ideal of X, then μ_ε is both an $(\in, \in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal and a $(q_0^\delta, \in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal of X.

Proof. Assume that L is a left ideal of X. Let $x, y, z \in X$ $z_t \in \mu_\varepsilon$ for $t \in (0, \delta]$. Then $\mu_\varepsilon(z) = \varepsilon$, and so $z \in L$. Thus $xyz \in L$, which implies that $\mu_\varepsilon(xyz) = \varepsilon$. If $t \leq \frac{\delta}{2}$, then $\mu_\varepsilon(xyz) = \varepsilon \geq \frac{\delta}{2} \geq t$, that is, $(xyz)_t \in \mu_\varepsilon$. If $t > \frac{\delta}{2}$, then $\mu_\varepsilon(xyz) + t = \varepsilon + t > \frac{\delta}{2} + \frac{\delta}{2} = \delta$ and so $(xyz)_t\, q_0^\delta\, \mu_\varepsilon$. Therefore $(xyz)_t \in\vee q_0^\delta \mu_\varepsilon$, and μ_ε is an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal of X. Now let $z \in X$ and $t \in (0, \delta]$ be such that $z_t\, q_0^\delta\, \mu_\varepsilon$. Then $\mu_\varepsilon(z) + t > \delta$ and so $\mu_\varepsilon(z) = \varepsilon$, that is, $z \in L$. Since L is a left ideal of X, it follows that $xyz \in L$ for all $x, y \in X$. Hence $\mu_\varepsilon(xyz) = \varepsilon \geq \frac{\delta}{2} \geq t$ whenever $t \leq \frac{\delta}{2}$. If $t > \frac{\delta}{2}$, then $\mu_\varepsilon(xyz) + t = \varepsilon + t > \frac{\delta}{2} + \frac{\delta}{2} = \delta$ and thus $(xyz)_t\, q_0^\delta\, \mu_\varepsilon$. Consequently, $(xyz)_t \in\vee q_0^\delta$, and therefore μ_ε is a $(q_0^\delta, \in\vee q_0^\delta)$-fuzzy left ideal of X. Similarly, we can prove that μ_ε is both an $(\in, \in\vee q_0^\delta)$-fuzzy right (resp., lateral) ideal of X and a $(q_0^\delta, \in\vee q_0^\delta)$-fuzzy right (resp., lateral) ideal of X whenever L is a right (resp., lateral) ideal. □

Corollary 3.18. *For a subset L of X, consider a fuzzy set μ_ε in X as follows:*

$$\mu_\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} \varepsilon \in [0.5, 1] & \text{if } x \in L, \\ 0 & \text{if } x \notin L. \end{cases}$$

If L is a left (resp., right, lateral) ideal of X, then μ_ε is both an $(\in, \in\vee q)$-fuzzy left (resp., right, lateral) ideal and a $(q, \in\vee q)$-fuzzy left (resp., right, lateral) ideal of X.

Theorem 3.19. *If the fuzzy set μ_ε in Theorem 3.17 is an $(\in, \in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal of X, then L is a left (resp., right, lateral) ideal of X.*

Proof. Assume that μ_ε is an $(\in, \in\vee q_0^\delta)$-fuzzy right ideal of X. Let $x \in LSS$ Then $x = ayz$ for some $a \in L$ and $y, z \in X$. Then

$$\mu_\varepsilon(x) = \mu_\varepsilon(ayz) \geq \min\{\mu_\varepsilon(a), \tfrac{\delta}{2}\} = \min\{\varepsilon, \tfrac{\delta}{2}\} = \tfrac{\delta}{2}, \tag{3.6}$$

and so $\mu_\varepsilon(x) = \varepsilon$. Hence $x \in L$ and L is a right ideal. Similarly, we can prove L is a left (lateral) ideal of X. □

Theorem 3.20. *A fuzzy set μ in X is an $(\in, \in)$-fuzzy left (resp., right, lateral) ideal of X if and only if the set*

$$Q_0^\delta(\mu; t) := \{x \in X \mid x_t\, q_0^\delta\, \mu\}$$

is a left (resp., right, lateral) ideal of X *for all* $t \in (0, \delta]$ *with* $Q_0^\delta(\mu; t) \neq \emptyset$.

Proof. Let $z \in Q_0^\delta(\mu; t)$ for $t \in (0, \delta]$. Then $z_t\, q_0^\delta\, \mu$, that is, $\mu(z) + t > \delta$. It follows that $\mu(xyz) + t \geq \mu(z) + t > \delta$ for all $x, y \in X$ whenever μ is an $(\in, \in)$-fuzzy left ideal of X. Hence $(xyz)_t\, q_0^\delta\, \mu$, and thus $xyz \in Q_0^\delta(\mu; t)$. Thus $Q_0^\delta(\mu; t)$ is a left ideal of X.

Conversely, suppose that the set

$$Q_0^\delta(\mu; t) := \{x \in X \mid x_t\, q_0^\delta\, \mu\}$$

is a left ideal of X for all $t \in (0, \delta]$ with $Q_0^\delta(\mu; t) \neq \emptyset$. Taking $a, b, c \in X$ and $t \in (0, \delta]$ such that

$$\mu(abc) + t \leq \delta < \mu(c) + t.$$

Then $c \in Q_0^\delta(\mu; t)$, and so $abc \in Q_0^\delta(\mu; t)$. Thus $\mu(abc) + t > \delta$, a contradiction. Hence $\mu(xyz) \geq \mu(z)$ for all $x, y, z \in X$, that is, μ is an $(\in, \in)$-fuzzy left ideal of X. Similarly we can prove the cases of right ideal and lateral ideal of X. □

Theorem 3.21. *For a fuzzy set* μ *in* X*, if the set* $Q_0^\delta(\mu; t)$ *is a left (resp., right, lateral) ideal of* X *for all* $t \in (\frac{\delta}{2}, 1]$*, then* μ *is a* $\frac{\delta}{2}$*-upper* $(\in, q_0^\delta)$*-fuzzy left (resp., right, lateral) ideal of* X.

Proof. Let $z_t \in \mu$ for $t \in (\frac{\delta}{2}, 1]$. Then $\mu(z) \geq t$, and so $\mu(z) + t > \frac{\delta}{2} + \frac{\delta}{2} = \delta$. Hence $z_t\, q_0^\delta\, \mu$, that is, $z \in Q_0^\delta(\mu; t)$. It follows that $xyz \in Q_0^\delta(\mu; t)$ for all $x, y \in X$ when $Q_0^\delta(\mu; t)$ is a left ideal of X. Thus $(xyz)_t\, q_0^\delta\, \mu$ and therefore μ is a $\frac{\delta}{2}$-upper $(\in, q_0^\delta)$-fuzzy left ideal of X whenever $Q_0^\delta(\mu; t)$ is a left ideal of X for all $t \in (\frac{\delta}{2}, 1]$. Similar way induces the desired results for the case of right and lateral ideal. □

Corollary 3.22 ([13])**.** *For a fuzzy set* μ *in* X*, if the set*

$$Q(\mu; t) := \{x \in X \mid x_t\ q\ \mu\}$$

is a left (resp., right, lateral) ideal of X *for all* $t \in (0.5, 1]$*, then* μ *is a* 0.5*-upper* $(\in, q)$*-fuzzy left (resp., right, lateral) ideal of* X.

Theorem 3.23. *For a fuzzy set* μ *in* X*, if the set* $Q_0^\delta(\mu; t)$ *is a left (resp., right, lateral) ideal of* X *for all* $t \in (0, \frac{\delta}{2}]$*, then* μ *is a* $\frac{\delta}{2}$*-lower* $(q_0^\delta, \in)$*-fuzzy left (resp., right, lateral) ideal of* X.

Proof. Assume that the set $Q_0^\delta(\mu; t)$ is a lateral ideal of X for all $t \in (0, \frac{\delta}{2}]$. Let $y_t\, q_0^\delta\, \mu$ for $y \in X$ and $t \in (0, \frac{\delta}{2}]$. Then $y \in Q_0^\delta(\mu; t)$ which implies $xyz \in Q_0^\delta(\mu; t)$ for all $x, z \in X$ because $Q_0^\delta(\mu; t)$ is a lateral ideal of X. Hence $(xyz)_t\, q_0^\delta\, \mu$, and so $\mu(xyz) + t > \delta$, that is, $\mu(xyz) > \delta - t \geq t$. Thus $(xyz)_t \in \mu$, and μ is a $\frac{\delta}{2}$-lower $(q_0^\delta, \in)$-fuzzy lateral ideal of X. Similarly, we can prove that μ is a $\frac{\delta}{2}$-lower $(q_0^\delta, \in)$-fuzzy left (right) ideal of X whenever the set $Q_0^\delta(\mu; t)$ is a left (right) ideal of X for all $t \in (0, \frac{\delta}{2}]$. □

Corollary 3.24. *For a fuzzy set* μ *in* X*, if the set* $Q_0^\delta(\mu; t)$ *is a left (resp., right, lateral) ideal of* X *for all* $t \in (0, 1]$*, then* μ *is both a* $\frac{\delta}{2}$*-upper* $(\in, q_0^\delta)$*-fuzzy left (resp., right, lateral) ideal and a* $\frac{\delta}{2}$*-lower* $(q_0^\delta, \in)$*-fuzzy left (resp., right, lateral) ideal of* X.

Corollary 3.25 ([13])**.** *For a fuzzy set* μ *in* X*, if the set* $Q(\mu; t)$ *is a left (resp., right, lateral) ideal of* X *for all* $t \in (0, 0.5]$*, then* μ *is a* 0.5*-lower* $(q, \in)$*-fuzzy left (resp., right, lateral) ideal of* X.

Theorem 3.26. *If* μ *is a* $(q_0^\delta, \in \vee q_0^\delta)$*-fuzzy left (resp., right, lateral) ideal of* X*, then the set* $Q_0^\delta(\mu; t)$ *is a left (resp., right, lateral) ideal of* X *when it is nonempty for all* $t \in (\frac{\delta}{2}, 1]$.

Proof. Assume that μ is a $(q_0^\delta, \in \vee q_0^\delta)$-fuzzy left ideal of X and let $t \in (\frac{\delta}{2}, 1]$ such that $Q_0^\delta(\mu; t) \neq \emptyset$. Let $z \in Q_0^\delta(\mu; t)$. Then $z_t\, q_0^\delta\, \mu$, and thus $(xyz)_t \in\vee q_0^\delta \mu$, that is, $(xyz)_t \in \mu$ or $(xyz)_t\, q_0^\delta\, \mu$ for all $x, y \in X$. If $(xyz)_t\, q_0^\delta\, \mu$, then $xyz \in Q_0^\delta(\mu; t)$. If $(xyz)_t \in \mu$, then $\mu(xyz) \geq t > \delta - t$ and so $(xyz)_t\, q_0^\delta\, \mu$. Hence $xyz \in Q_0^\delta(\mu; t)$, and therefore $Q_0^\delta(\mu; t)$ is a left ideal of X. Similar way leads to the proof of other cases. □

Corollary 3.27 ([13])**.** *If μ is a $(q, \in \vee\, q)$-fuzzy left (resp., right, lateral) ideal of X, then the set $Q(\mu; t)$ is a left (resp., right, lateral) ideal of X when it is nonempty for all $t \in (0.5, 1]$.*

Theorem 3.28. *For any fuzzy set μ in X, if the set*

$$X^t_{\in\vee q_0^\delta} := \{x \in X \mid x_t \in\vee q_0^\delta\, \mu\}$$

is a left (resp., right, lateral) ideal of X for all $t \in (0, \delta]$ with $X^t_{\in\vee q_0^\delta} \neq \emptyset$, then μ is a $(q_0^\delta$, $\in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal of X.

Proof. Let $z\, q_0^\delta\, \mu$ for $t \in (0, \delta]$. Then $z \in Q_0^\delta(\mu; t) \subseteq X^t_{\in\vee q_0^\delta}$, and hence $xyz \in X^t_{\in\vee q_0^\delta}$ for all $x, y \in X$ when $X^t_{\in\vee q_0^\delta}$ is a left ideal of X. It follows that $(xyz)_t \in\vee q_0^\delta \mu$ and so that μ is a $(q_0^\delta, \in\vee q_0^\delta)$-fuzzy left ideal of X whenever $X^t_{\in\vee q_0^\delta}$ is a left ideal of X. By the similar way, we can induce other cases. □

Corollary 3.29 ([13])**.** *For any fuzzy set μ in X, if the set*

$$X^t_{\in\vee\, q} := \{x \in X \mid x_t \in\vee\, q\, \mu\}$$

is a left (resp., right, lateral) ideal of X for all $t \in (0, 1]$ with $X^t_{\in\vee\, q} \neq \emptyset$, then μ is a $(q$, $\in\vee\, q)$-fuzzy left (resp., right, lateral) ideal of X.

We have the following question.

Question 1. *If μ is an $(\in, \in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal of X, then is the set $Q_0^\delta(\mu; t)$ a left (resp., right, lateral) ideal of X?*

The answer to the above question is negative for $t \leq \frac{\delta}{2}$ as seen in the following example.

Example 3.4. Consider the $(\in, \in \vee q_0^\delta)$-fuzzy left ideal μ of X for $\delta = 0.8$ in Example 3.2. Then the set

$$Q_0^{0.8}(\mu; 0.3) = \{x \in X \mid x_{0.3}\, q_0^{0.8}\, \mu\} = \{a, b\}$$

is not a left ideal of X since $aab = d \notin Q_0^{0.8}(\mu; 0.3)$.

Given an $(\in, \in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal of X, we now provide conditions for the set $Q_0^\delta(\mu; t)$ to be a left (resp., right, lateral) ideal of X.

Theorem 3.30. *Given an $(\in, \in \vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal μ of X, if $t \in (\frac{\delta}{2}, 1]$ then the set $Q_0^\delta(\mu; t)$ is a left (resp., right, lateral) ideal of X.*

Proof. Let $z \in Q_0^\delta(\mu; t)$ for $t \in (\frac{\delta}{2}, 1]$. Then $\mu(z) + t > \delta$, and so

$$\mu(xyz) + t \geq \min\{\mu(z), \tfrac{\delta}{2}\} + t = \min\{\mu(z) + t, \tfrac{\delta}{2} + t\} > \delta$$

for all $x, y \in X$ whenever μ is an $(\in, \in \vee q_0^\delta)$-fuzzy left ideal of X. Hence $(xyz)_t\, q_0^\delta\, \mu$, that is, $xyz \in Q_0^\delta(\mu; t)$. Therefore $Q_0^\delta(\mu; t)$ is a left ideal of X. The same argument leads to the proof of the cases of right and lateral ideal. □

Corollary 3.31 ([13])**.** *Given an $(\in, \in\vee\, q)$-fuzzy left (resp., right, lateral) ideal μ of X, if $t \in (0.5, 1]$ then the set $Q(\mu; t)$ is a left (resp., right, lateral) ideal of X.*

Theorem 3.32. *For a fuzzy set μ in X, the following are equivalent.*

(i) *μ is an $(\in, \in\vee q_0^\delta)$-fuzzy left (resp., right, lateral) ideal of X.*
(ii) *The set $X^t_{\in\vee q_0^\delta}$ is a left (resp., right, lateral) ideal of X for all $t \in (0, \delta]$.*

Proof. Assume that μ is an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal of X. Then

$$\mu(xyz) \geq \min\{\mu(z), \tfrac{\delta}{2}\} \tag{3.7}$$

for all $x, y, z \in X$. Let $z \in X^t_{\in\vee q_0^\delta}$ for $t \in (0, \delta]$. Then $\mu(z) \geq t$ or $\mu(z) + t > \delta$. We first consider the case $\mu(z) \geq t$. If $t > \frac{\delta}{2}$, then

$$\mu(xyz) \geq \min\{\mu(z), \tfrac{\delta}{2}\} \geq \min\{t, \tfrac{\delta}{2}\} = \tfrac{\delta}{2} \tag{3.8}$$

and so $\mu(xyz) + t \geq \frac{\delta}{2} + t > \delta$, that is, $(xyz)_t\, q_0^\delta\, \mu$. If $t \leq \frac{\delta}{2}$, then

$$\mu(xyz) \geq \min\{\mu(z), \tfrac{\delta}{2}\} \geq \min\{t, \tfrac{\delta}{2}\} = t \tag{3.9}$$

and thus $(xyz)_t \in \mu$. Hence $(xyz)_t \in\vee q_0^\delta \mu$, that is, $xyz \in X^t_{\in\vee q_0^\delta}$. Now, we consider the case $\mu(z) + t > \delta$. If $t > \frac{\delta}{2}$, then $\delta - t < \frac{\delta}{2} < t$ and

$$\begin{aligned} \mu(xyz) &\geq \min\{\mu(z), \tfrac{\delta}{2}\} \\ &= \begin{cases} \mu(z) & \text{if } \mu(z) < \frac{\delta}{2}, \\ \frac{\delta}{2} & \text{if } \mu(z) \geq \frac{\delta}{2}, \end{cases} \\ &> \delta - t. \end{aligned} \tag{3.10}$$

Thus $(xyz)_t\, q_0^\delta\, \mu$. If $t \leq \frac{\delta}{2}$, then

$$\mu(xyz) \geq \min\{\mu(z), \tfrac{\delta}{2}\} \geq \min\{\delta - t, \tfrac{\delta}{2}\} = \tfrac{\delta}{2} \geq t \tag{3.11}$$

and so $xyz \in U(\mu; t) \subseteq X^t_{\in\vee q_0^\delta}$. Therefore $X^t_{\in\vee q_0^\delta}$ is a left ideal of X. Similarly we can prove that $X^t_{\in\vee q_0^\delta}$ is a right (lateral) ideal of X whenever μ is an $(\in, \in\vee q_0^\delta)$-fuzzy right (lateral) ideal of X.

Conversely, suppose that $X^t_{\in\vee q_0^\delta}$ is a lateral ideal of X for $t \in (0, \delta]$. If possible, let

$$\mu(abc) < t \leq \min\{\mu(b), \tfrac{\delta}{2}\} \tag{3.12}$$

for some $a, b, c \in X$ and $t \in (0, \frac{\delta}{2}]$. Then $b \in U(\mu; t) \subseteq X^t_{\in\vee q_0^\delta}$, which implies that $abc \in X^t_{\in\vee q_0^\delta}$ since $X^t_{\in\vee q_0^\delta}$ is a lateral ideal of X. It follows that $\mu(abc) \geq t$ or $\mu(abc) + t > \delta$, a contradiction. Therefore $\mu(xyz) \geq \min\{\mu(y), \frac{\delta}{2}\}$ for all $x, y, z \in X$. It follows from Theorem 3.3 that μ is an $(\in, \in\vee q_0^\delta)$-fuzzy lateral ideal of X. Similarly, we can show that μ is an $(\in, \in\vee q_0^\delta)$-fuzzy left (right) ideal of X whenever $X^t_{\in\vee q_0^\delta}$ is a left (right) ideal of X. □

Corollary 3.33 ([13])**.** *A fuzzy set μ in X is an $(\in, \in\vee q)$-fuzzy left (resp., right, lateral) ideal of X if and only if the set $X^t_{\in\vee q}$ is a left (resp., right, lateral) ideal of X for all $t \in (0, 1]$.*

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