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m-POLAR CUBIC SET THEORY APPLIED TO *BCK/BCI*-ALGEBRAS

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ABSTRACT. In this paper, by combinig the notions of m-polar fuzzy structures and interval valued m-polar fuzzy structures, the notion of m-polar cubic structures is introduced and applied on the ideal theory of BCK/BCI-algebras. In this respect, the notions of m-polar cubic subalgebras and m-polar cubic (commutative) ideals are introduced and some essential properties are discussed. Characterizations of m-polar cubic subalgebras and m-polar cubic (commutative) ideals are considered. Moreover, the relations among m-polar cubic subalgebras, m-polar cubic ideals and m-polar cubic commutative ideals are obtained.

1. INTRODUCTION

Imai and Iséki presented the *BCK/BCI*-algebras [13, 14] in 1966, which is an extension of set-theoretic difference and propositional calculus. Since then, a lot of research has emerged on the theory of *BCK/BCI*-algebras, with a particular focus on the ideal theory of *BCK/BCI*-algebras. Different types of ideals were examined in various methods in BCK/BCI-algebras (see, for example, [15, 16, 27, 28, 31]).

By combining the notions of fuzzy sets and interval valued fuzzy sets, Jun et al. [21] introduced the notion of cubic sets (see [22, 23, 24] for related ideas and results on cubic ideals in *BCK/BCI*-algebras). Thereafter, the notion of cubic ideals was introduced in different algebraic structures and studied by several authors, for instance, Muhiuddin et al. [32, 33], Senapati et al. [38, 39, 40], Gaketem et al. [11], Gulistan [12], Yaqoob et al. [42], and many others.

In 2014, Chen et al. [8] presented the m-polar fuzzy set, an expansion of the bipolar fuzzy set. The m-polar fuzzy models provide the framework with more accuracy, versatility and compatibility when more than one varibles needs to be taken. The m-polar fuzzy algebraic structures study began with the concept of *m-pF* lie subalgebras introduced by Akram et al. [1]. After that, the theory of *m*-pF lie ideals was introduced by Akram et al. [1] in lie subalgebras. A concept given by [10] for the *m-pF* subgroups. Al-Masarwah et. al [3] proposed the concepts of *m-pF* ideals and *m-pF* commutative ideals on *BCK/BCI*-algebras. To make this paper self-readable, readers are suggested to read [6, 34, 35, 36, 37].

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In this paper, by combinig the notions of m-polar fuzzy sets and interval valued m-polar fuzzy sets, the notion of m-polar cubic structures is introduced and applied on the ideal theory of BCK/BCI-algebra. In this respect, we introduce the notions of m-polar cubic subalgebra, m-polar cubic ideals and m-polar cubic commutative ideals. We prove that m-polar cubic ideals are m-polar cubic subalgebrsa but the converse statement is not valid and an example is given in this support. We provide a condition under which m-polar cubic subalgebra becomes an m-polar cubic ideals. Moreover, we prove that m-polar cubic commutative ideals are m-polar cubic ideals but the converse implication is not true and an example is given in this aim. Also, a condition under which m-polar cubic ideal becomes an m-polar cubic commutative ideal is provided.

2. PRELIMINARIES

An algebra $(\widetilde{\mathcal{A}}; *, 0)$ of type $(2, 0)$ is said to be a ***BCI-algebra*** if:
(K_1) $((\vartheta * \ell) * (\vartheta * \varrho)) * (\varrho * \ell) = 0$,
(K_2) $(\vartheta * (\vartheta * \ell)) * \ell = 0$,
(K_3) $\vartheta * \vartheta = 0$,
(K_4) $\vartheta * \ell = 0$ and $\ell * \vartheta = 0 \Rightarrow \vartheta = \ell$,
$\forall\, \vartheta, \varrho, \ell \in \widetilde{\mathcal{A}}$.
If a BCI-algebra $\widetilde{\mathcal{A}}$ satisfies the condition:
(K_5) $0 * \vartheta = 0, \forall\, \vartheta \in \widetilde{\mathcal{A}}$,
then $\widetilde{\mathcal{A}}$ is a BCK-algebra, .

Any BCK/BCI-algebra $\widetilde{\mathcal{A}}$ has the following properties:
(π_1) $\vartheta * 0 = \vartheta$,
(π_2) $(\vartheta * \ell) * \varrho = (\vartheta * \varrho) * \ell$,
(π_3) $\vartheta \leq \ell \Rightarrow \vartheta * \varrho \leq \ell * \varrho$ and $\varrho * \ell \leq \varrho * \vartheta$,
(π_4) $0 * (\vartheta * \ell) = (0 * \vartheta) * (0 * \ell)$,
(π_5) $0 * (0 * (\vartheta * \ell)) = 0 * (\ell * \vartheta)$,
(π_6) $(\vartheta * \varrho) * (\ell * \varrho) \leq (\vartheta * \ell)$,
(π_7) $\vartheta * (\vartheta * (\vartheta * \ell)) = \vartheta * \ell$,
(π_8) $0 * (0 * ((\vartheta * \varrho) * (\ell * \varrho))) = (0 * \ell) * (0 * \vartheta)$,
(π_9) $0 * (0 * (\vartheta * \ell) = (0 * \ell) * (0 * \vartheta)$,
where $\vartheta \leq \ell \Leftrightarrow \vartheta * \ell = 0\ \forall\, \vartheta, \varrho, \ell \in \widetilde{\mathcal{A}}$. Note that $(\widetilde{\mathcal{A}}, \leq)$ is a partially ordered set.

A set $Z(\neq \emptyset)$ of $\widetilde{\mathcal{A}}$ is said to be a $subalgebra$ of $\widetilde{\mathcal{A}}$ if $\vartheta * \ell \in Z\ \forall\, \vartheta, \ell \in \widetilde{\mathcal{A}}$ and it is called an $ideal$ of Z if $0 \in Z$ and $\forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}}, \vartheta * \varrho \in Z, \varrho \in Z$ implies $\vartheta \in Z$. Further, Z is called commutative ideal of $\widetilde{\mathcal{A}}$ if $0 \in Z$ and $\forall\, \vartheta, \varrho, \omega \in Z, ((\vartheta * \omega) * (\varrho * \omega)) \in Z, \varrho \in Z$ implies $\vartheta \in Z$.

We mean an interval defined by $[r^-, r^+]$ where $0 \leq r^- \leq r^+ \leq 1$ by an interval number $\tilde{r}$. $D[0, 1]$ denotes the set of all interval numbers. For the intervals $[r_\imath^-, r_\imath^+]$, $[s_\imath^-, s_\imath^+] \in D[0, 1], \imath \in I$, we define

(a) $\min\{[r_\imath^-, r_\imath^+], [s_\imath^-, s_\imath^+]\} = [\min(r_\imath^-, s_\imath^-), \min(r_\imath^+, s_\imath^+)]$;
(b) $\max\{[r_\imath^-, r_\imath^+], [s_\imath^-, s_\imath^+]\} = [\max(r_\imath^-, s_\imath^-), \max(r_\imath^+, s_\imath^+)]$;
(c) $[r_\imath^-, r_\imath^+] \leq [s_\imath^-, s_\imath^+] \Leftrightarrow r_\imath^- \leq s_\imath^-$ and $r_\imath^+ \leq s_\imath^+$;
(d) $[r_\imath^-, r_\imath^+] = [s_\imath^-, s_\imath^+] \Leftrightarrow r_\imath^- = s_\imath^-$ and $r_\imath^+ = s_\imath^+$.

Assume that $\widetilde{\mathcal{A}}$ is a *BCK/BCI*-algebra. A mapping $\widetilde{\Psi^P} : \widetilde{\mathcal{A}} \to D[0,1]$ is referred to as an interval valued fuzzy set (IVFS) in $\widetilde{\mathcal{A}}$, where $\widetilde{\Psi^P}(\vartheta) = [\widetilde{\Psi^P}^-(\vartheta), \widetilde{\Psi^P}^+(\vartheta)]\ \forall\ \vartheta \in \widetilde{\mathcal{A}}$, $\widetilde{\Psi^P}^-$ and $\widetilde{\Psi^P}^+$ are fuzzy sets of $\widetilde{\mathcal{A}}$ with $\widetilde{\Psi^P}^-(\vartheta) \leq \widetilde{\Psi^P}^+(\vartheta), \forall\ \vartheta \in \widetilde{\mathcal{A}}$.

Definition 2.1. A cubic set $\mathcal{C_S}$ on $\widetilde{\mathcal{A}}$ is a structure

$$\mathcal{C_S} = \{(\vartheta, \widetilde{\Psi^P}(\vartheta), \widetilde{\Phi^P}(\vartheta)) \mid \vartheta \in \widetilde{\mathcal{A}}\}$$

which is denoted by $\mathcal{C_S} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$, where $\widetilde{\Psi^P}$ is an IVFS and $\widetilde{\Phi^P}$ is a FS in $\widetilde{\mathcal{A}}$.

Definition 2.2. A mapping $\widetilde{\Phi^P} : \widetilde{\mathcal{A}} \to [0,1]^m$ is reffered to as an *m*-polar fuzzy set (*mpF* set) of $\widetilde{\mathcal{A}}$ and is described as:

$$\widetilde{\Phi^P}(\vartheta) = (\widetilde{\varpi_1} \circ \widetilde{\Phi^P}(\vartheta), \widetilde{\varpi_2} \circ \widetilde{\Phi^P}(\vartheta), \ldots, \widetilde{\varpi_m} \circ \widetilde{\Phi^P}(\vartheta))$$

where $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta)$ represents the i-th degree of membership of ϑ.

Define an order " $\leq$ " on $[0,1]^m$ as pointwise i.e.,

$$\vartheta \leq \varrho \Leftrightarrow \widetilde{\varpi_\imath}(\vartheta) \leq \widetilde{\varpi_\imath}(\varrho)\ \forall\ 1 \leq \imath \leq m.$$

The $\imath$-th projection mapping is represented as $\widetilde{\varpi_\imath} : [0,1]^m \to [0,1]$. We mean $(\ell, \ell, ..., \ell)$ by $\widetilde{\ell} \in [0,1]^m$. Thus, the smallest and greatest elements in $[0,1]^m$ are $\widetilde{0}$ and $\widetilde{1}$.

Definition 2.3. A mapping $\widetilde{\Psi^P} : \widetilde{\mathcal{A}} \to D[0,1]^m$ is reffered to as an interval valued *m*-polar fuzzy set (*IVmPF* set) of $\widetilde{\mathcal{A}}$ and is described as:

$$\widetilde{\Psi^P}(\vartheta) = (\widetilde{\varpi_1} \circ \widetilde{\Psi^P}(\vartheta), \widetilde{\varpi_2} \circ \widetilde{\Psi^P}(\vartheta), \ldots, \widetilde{\varpi_m} \circ \widetilde{\Psi^P}(\vartheta)),$$

where $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}$ represents the i-th degree of membership of ϑ.
That is
$\widetilde{\Psi^P}(\vartheta) = ([\Psi^{P}{}^-_1(\vartheta), \Psi^{P}{}^+_1(\vartheta)], [\Psi^{P}{}^-_2(\vartheta), \Psi^{P}{}^+_2(\vartheta)], \ldots, [\Psi^{P}{}^-_m(\vartheta), \Psi^{P}{}^+_m(\vartheta)]), \forall\ \vartheta \in \widetilde{\mathcal{A}}$
where $\Psi^{P}{}^-_\imath$ and $\Psi^{P}{}^+_\imath$ are fuzzy sets of $\widetilde{\mathcal{A}}$ with $\Psi^{P}{}^-_\imath(\vartheta) \leq \Psi^{P}{}^+_\imath(\vartheta)$, $\forall\ \vartheta \in \widetilde{\mathcal{A}}$ and $1 \leq \imath \leq m$.

On $D[0,1]^m$, a pointwise order is defined as follows:

$$\vartheta \leq \varrho \Leftrightarrow \widetilde{\varpi_\imath}(\vartheta) \leq \widetilde{\varpi_\imath}(\varrho),\ \forall\ 1 \leq \imath \leq m.$$

The $\jmath$-th projection mapping is represented as $\widetilde{\varpi_\imath} : D[0,1]^m \to D[0,1]$. We mean $\{[\Theta, \theta], [\Theta, \theta], ..., [\Theta, \theta]\}$ when we say $\widetilde{[\Theta, \theta]} \in D[0,1]^m$. Thus, the smallest and greatest elements in $D[0,1]^m$ are $\widetilde{[0,0]}$ and $\widetilde{[1,1]}$.

3. m-POLAR CUBIC SUBALGEBRAS

mpC subalgebras in *BCK/BCI*-algebras are described and characterised in this section.

Definition 3.1. Let $\widetilde{\mathcal{A}}$ be a *BCK/BCI*-algebra. An *m*-polar cubic set $\mathcal{C_S}$ (briefly, $mpCS$) is a structure

$$\mathcal{C_S} = \{(\vartheta, \widetilde{\Psi^P}(\vartheta), \widetilde{\Phi^P}(\vartheta)) \mid \vartheta \in \widetilde{\mathcal{A}}\}$$

which is denoted by $\mathcal{C_S} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$, where $\widetilde{\Psi^P}$ is an *IVmPFS* and $\widetilde{\Phi^P}$ is an *mpFS* in $\widetilde{\mathcal{A}}$.

Definition 3.2. An $mpCS$ $\mathcal{C_S} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ of $\widetilde{\mathcal{A}}$ is called an *mpC* subalgebra (briefly, *mpCSub*) if:

(C1) $(\forall\ \vartheta, \varrho \in \widetilde{\mathcal{A}})\ \widetilde{\Psi^P}(\vartheta * \varrho) \geq \widetilde{\Psi^P}(\vartheta) \wedge \widetilde{\Psi^P}(\varrho)$, (C2) $(\forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}})\ \widetilde{\Phi^P}(\vartheta * \varrho) \leq \widetilde{\Phi^P}(\vartheta) \vee \widetilde{\Phi^P}(\varrho)$,

that is,

(C1) $\big(\forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}},\ 1 \leq \imath \leq m\big)\ \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)$,

(C2) $\big(\forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}},\ 1 \leq \imath \leq m\big)\ \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho)$.

Example 3.3. Conseder a BCK-algebra $\widetilde{\mathcal{A}} = \{0, \vartheta, \varrho, \ell\}$ with the following table.

$*$	0	ϑ	ϱ	ℓ
0	0	0	0	0
ϑ	ϑ	0	0	ϑ
ϱ	ϱ	ϑ	0	ϱ
ℓ	ℓ	ℓ	ℓ	0

TABLE 1. Cayley table for $*$-operation

Let $\widehat{[\omega, \varphi]} = \big([\omega_1, \varphi_1], [\omega_2, \varphi_2], \ldots, [\omega_m, \varphi_m]\big), \widehat{[\Theta, \theta]} = \big([\Theta_1, \theta_1], [\Theta_2, \theta_2], \ldots, [\Theta_m, \theta_m]\big) \in D[0,1]^m$ and $\widehat{\jmath} = (\jmath_1, \jmath_2, ..., \jmath_m), \widehat{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) \in [0,1]^m$ be such that $\widehat{[\omega, \varphi]} \geq \widehat{[\Theta, \theta]}$ and $\widehat{\jmath} \geq \widehat{\varepsilon}$. Now define an *mpCS* $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ on $\widetilde{\mathcal{A}}$ as:

$*$	$\widetilde{\Psi^P}$	$\widetilde{\Phi^P}$
0	$\widehat{[\omega, \varphi]} = \big([\omega_1, \varphi_1], [\omega_2, \varphi_2], \ldots, [\omega_m, \varphi_m]\big)$	$(0, 0, ..., 0)$
ϑ	$\widehat{[\Theta, \theta]} = \big([\Theta_1, \theta_1], [\Theta_2, \theta_2], \ldots, [\Theta_m, \theta_m]\big)$	$(0, 0, ..., 0)$
ϱ	$\big([0,0], [0,0], \ldots, [0,0]\big)$	$\widehat{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m)$
ℓ	$\big([0,0], [0,0], \ldots, [0,0]\big)$	$\widehat{\jmath} = (\jmath_1, \jmath_2, ..., \jmath_m)$

TABLE 2. Table for the membership values

It is straightforward to show that $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is a 3*pCSub* of $\widetilde{\mathcal{A}}$.

Lemma 3.1. *If $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an mpCSub of $\widetilde{\mathcal{A}}$, then*

$$\widetilde{\Psi^P}(0) \geq \widetilde{\Psi^P}(\vartheta) \text{ and } \widetilde{\Phi^P}(0) \leq \widetilde{\Phi^P}(\vartheta)\ \forall\, \vartheta \in \widetilde{\mathcal{A}}.$$

Proof. Let $\vartheta \in \widetilde{\mathcal{A}}$. Then, we have

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(0) &= \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \vartheta)\\ &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta)\\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(0) &= \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \vartheta)\\ &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta)\\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta),\end{aligned}$$

as required. □

Definition 3.4. Let $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ be any *mpCS* . For $\widehat{[\Theta, \theta]} = \big([\Theta_1, \theta_1], [\Theta_2, \theta_2], \ldots,$ $\big) \in D[0,1]^m$ and $\hat{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) \in [0,1]^m$ define a level set $U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$ as follows:

$U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \widehat{\varepsilon}) = \{x \in \widetilde{\mathcal{A}} \mid \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(x) \geq [\Theta_\imath, \theta_\imath] \text{ and } \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(x) \leq \varepsilon_\imath\ \forall\, 1 \leq \imath \leq m\}.$

Theorem 3.2. *An mpCS* $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ *is an mpCSub of* $\widetilde{\mathcal{A}} \Leftrightarrow$ *each* $(\emptyset \neq) U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$ *is a subalgebra of* $\widetilde{\mathcal{A}}$, $\forall$ $\widehat{[\Theta, \theta]} = \big([\Theta_1, \theta_1], [\Theta_2, \theta_2], \dots, [\Theta_m, \theta_m]\big) \in D[0,1]^m$ *and* $\hat{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) \in [0,1]^m$.

Proof. ($\Rightarrow$) Take any $\vartheta, \varrho \in U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$. Therefore $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \geq [\Theta_\imath, \theta_\imath]$, $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \leq \varepsilon_\imath$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho) \geq [\Theta_\imath, \theta_\imath]$, $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho) \leq \varepsilon_\imath$. As $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an *mpCSub* of $\widetilde{\mathcal{A}}$, so we have

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\\ &\geq [\Theta_\imath, \theta_\imath] \wedge [\Theta_\imath, \theta_\imath]\\ &= [\Theta_\imath, \theta_\imath]\end{aligned}$$

and

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho)\\ &\leq \varepsilon_\imath \vee \varepsilon_\imath\\ &= \varepsilon_\imath.\end{aligned}$$

Therefore $\vartheta * \varrho \in U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$.

($\Leftarrow$) Assume that $U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$ is subalgebra of $\widetilde{\mathcal{A}}$, $\forall$ $\widehat{[\Theta, \theta]} = \big([\Theta_1, \theta_1], [\Theta_2, \theta_2], \dots, [\Theta_m, \theta_m]\big) \in D[0,1]^m$ and $\hat{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) \in [0,1]^m$. On contrary, let $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) < \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) > \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho)$ for some $\vartheta, \varrho \in \widetilde{\mathcal{A}}$. So there exist $\widehat{[\delta, \gamma]} = ([\delta_1, \gamma_1], [\delta_2, \gamma_2], ..., [\delta_m, \gamma_m]) \in D[0,1]^m$ and $\hat{\ell} = (\ell_1, \ell_2, ..., \ell_m) \in [0,1]^m$ such that $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) < [\delta_\imath, \gamma_\imath] \leq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) > \ell_\imath \geq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho)$ for each $1 \leq \imath \leq m$ implies $\vartheta, \varrho \in U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$ but $\vartheta * \varrho \notin U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$, which is not possible. Therefore $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho)$, $\forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}}$ and $1 \leq \imath \leq m$. Hence $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an *mpCSub* of $\widetilde{\mathcal{A}}$. □

4. m-POLAR CUBIC IDEALS

In this section, the concept of *mpC* ideal in *BCK/BCI*-algebras is described, and associated properties of *mpC* ideals and *mpC* subalgebras are discussed.

Definition 4.1. An *mpCS* $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is called an *mpC* ideal (briefly, *mpCI*) if:

(C3) $(\forall\, \vartheta \in \widetilde{\mathcal{A}})\ \widetilde{\Psi^P}(0) \geq \widetilde{\Psi^P}(\vartheta)$ and $\widetilde{\Phi^P}(0) \leq \widetilde{\Phi^P}(\vartheta)$,
(C4) $(\forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}})\ \widetilde{\Psi^P}(\vartheta) \geq \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\Psi^P}(\varrho)$,
(C5) $(\forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}})\ \widetilde{\Phi^P}(\vartheta) \leq \widetilde{\Phi^P}(\vartheta * \varrho) \vee \widetilde{\Phi^P}(\varrho)$,

that is,

(C3) $(\forall\, \vartheta \in \widetilde{\mathcal{A}},\ 1 \leq \imath \leq m)\ \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(0) \geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta)$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(0) \leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta)$,
(C4) $\big(\forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}},\ 1 \leq \imath \leq m\big)\ \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)$,
(C5) $\big(\forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}},\ 1 \leq \imath \leq m\big)\ \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho)$.

Example 4.2. Consider a BCI-algebra $\widetilde{\mathcal{A}} = \{0, \vartheta, \varrho, \ell\}$ with the following table.

$*$	0	1	ϑ	ϱ	ℓ
0	0	0	ϑ	ϱ	ℓ
1	1	0	ϑ	ϱ	ℓ
ϑ	ϑ	ϑ	0	ℓ	ϱ
ϱ	ϱ	ϱ	ℓ	0	ϑ
ℓ	ℓ	ℓ	ϱ	ϑ	0

TABLE 3. Cayley table for $*$-operation

Now define an *3pC* set $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ on $\widetilde{\mathcal{A}}$ as:

$*$	$\widetilde{\Psi^P}$	$\widetilde{\Phi^P}$
0	$([0.6, 0.7], [0.5, 0.8], [0.3, 0.4])$	$(0.3, 0.1, 0.2)$
1	$([0.5, 0.6], [0.3, 0.5], [0.2, 0.3])$	$(0.3, 0.2, 0.1)$
ϑ	$([0.2, 0.4], [0.1, 0.2], [0.1, 0.2])$	$(0.3, 0.3, 0.2)$
ϱ	$([0.3, 0.4], [0.2, 0.3], [0.1, 0.2])$	$(0.6, 0.4, 0.2)$
ℓ	$([0.2, 0.4], [0.1, 0.2], [0.1, 0.2])$	$(0.6, 0.4, 0.2)$

TABLE 4. Table for the membership values

It is simple to show that $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is a *3pCI* of $\widetilde{\mathcal{A}}$.

Lemma 4.1. *Let $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ be an mpCI of $\widetilde{\mathcal{A}}$ and $\vartheta, \varrho \in \widetilde{\mathcal{A}}$ such that $\vartheta \leq \varrho$. Then*

$$\widetilde{\Psi^P}(\vartheta) \geq \widetilde{\Psi^P}(\varrho) \text{ and } \widetilde{\Phi^P}(\vartheta) \leq \widetilde{\Phi^P}(\varrho).$$

Proof. Let $\vartheta, \varrho \in \widetilde{\mathcal{A}}$ such that $\vartheta \leq \varrho$. Then we have

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(0) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho).\end{aligned}$$

and

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho)\\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(0) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho)\\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho).\end{aligned}$$

□

Lemma 4.2. *Let $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ be an mpCI of $\widetilde{\mathcal{A}}$ and $\vartheta, \varrho, \hbar \in \widetilde{\mathcal{A}}$ such that $\vartheta * \varrho \leq \hbar$. Then*

$$\widetilde{\Psi^P}(\vartheta) \geq \widetilde{\Psi^P}(\varrho) \wedge \widetilde{\Psi^P}(\hbar) \text{ and } \widetilde{\Phi^P}(\vartheta) \leq \widetilde{\Phi^P}(\varrho) \vee \widetilde{\Phi^P}(\hbar).$$

Proof. Let $\vartheta, \varrho, \hbar \in \widetilde{\mathcal{A}}$ such that $\vartheta * \varrho \leq \hbar$. Then, we have

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\\ &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}((\vartheta * \varrho) * \hbar) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\hbar) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}((\vartheta * \varrho) * \hbar) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\hbar) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(0) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\hbar) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\hbar) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho) \\ &\leq \{\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}((\vartheta * \varrho) * \hbar) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\hbar)\} \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho) \\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}((\vartheta * \varrho) * \hbar) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\hbar) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho) \\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(0) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\hbar) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho) \\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\hbar) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho).\end{aligned}$$

Hence $\widetilde{\Psi^P}(\vartheta) \geq \widetilde{\Psi^P}(\varrho) \wedge \widetilde{\Psi^P}(\hbar)$ and $\widetilde{\Phi^P}(\vartheta) \leq \widetilde{\Phi^P}(\varrho) \vee \widetilde{\Phi^P}(\hbar)$. □

Theorem 4.3. *Every mpCI of BCK-algebra $\widetilde{\mathcal{A}}$ is an $mpCSub$ of $\widetilde{\mathcal{A}}$.*

Proof. Let $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ be any *mpCI* and $\vartheta, \varrho \in \widetilde{\mathcal{A}}$. As $\vartheta * \varrho \leq \vartheta$ in $\widetilde{\mathcal{A}}$, so by above Lemma 4.1, $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \leq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho)$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \geq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho)$. Therefore, we have

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \\ &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho) \\ &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \\ &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho) \\ &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho).\end{aligned}$$

Hence $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an $mpCSub$ of $\widetilde{\mathcal{A}}$. □

Remark. *Converse of above Theorem is not true in general.*

Example 4.3. Consider a BCK-algebra $\widetilde{\mathcal{A}} = \{0, \vartheta, \varrho, \ell\}$ with the following table:

$*$	0	ϑ	ϱ	ℓ
0	0	0	0	0
ϑ	ϑ	0	ϑ	0
ϱ	ϱ	ϱ	0	0
ℓ	ℓ	ℓ	ℓ	0

TABLE 5. Cayley table for $*$-operation

Now define an *3pC* set $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ on $\widetilde{\mathcal{A}}$ as:

$*$	$\widetilde{\Psi^P}$	$\widetilde{\Phi^P}$
0	$([0.7, 0.8], [0.3, 0.5], [0.2, 0.3])$	$(0.3, 0.1, 0.2)$
ϑ	$([0.5, 0.6], [0.1, 0.3], [0.1, 0.2])$	$(0.3, 0.2, 0.3)$
ϱ	$([0.3, 0.4], [0.1, 0.1], [0.1, 0.1])$	$(0.3, 0.3, 0.2)$
ℓ	$([0.6, 0.7], [0.3, 0.4], [0.1, 0.3])$	$(0.6, 0.4, 0.2)$

TABLE 6. Table for the membership values

It is easy to verify that $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is a $3pCSub$ of $\widetilde{\mathcal{A}}$ but not a $3pCI$ of $\widetilde{\mathcal{A}}$ beacuse $[0.6, 0.5] = (\pi_1 \circ \widetilde{\Psi^P})(\vartheta) \ngeq \pi_1 \circ \widetilde{\Psi^P}(\vartheta * \ell) \wedge \pi_1 \circ \widetilde{\Psi^P}(\ell) = [0.7, 0.6]$.

Theorem 4.4. *Let $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ be an mpCSub of $\widetilde{\mathcal{A}}$. Then $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an mpCI $\Leftrightarrow \forall\ \vartheta, \varrho, \hbar \in \widetilde{\mathcal{A}}$ such that $\vartheta * \varrho \leq \hbar$ implies $\widetilde{\Psi^P}(\vartheta) \geq \widetilde{\Psi^P}(\varrho) \wedge \widetilde{\Psi^P}(\hbar)$ and $\widetilde{\Phi^P}(\vartheta) \leq \widetilde{\Phi^P}(\varrho) \vee \widetilde{\Phi^P}(\hbar)$.*

Proof. ($\Rightarrow$) Follows from Lemma 4.2.

($\Leftarrow$) Let $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ be an *mpCSub* of $\widetilde{\mathcal{A}}$ such that for all $\vartheta, \varrho, \hbar \in \widetilde{\mathcal{A}}$, $\vartheta * \varrho \leq \hbar$ implies $\widetilde{\Psi^P}(\vartheta) \geq \widetilde{\Psi^P}(\varrho) \wedge \widetilde{\Psi^P}(\hbar)$ and $\widetilde{\Phi^P}(\vartheta) \leq \widetilde{\Phi^P}(\varrho) \vee \widetilde{\Phi^P}(\hbar)$. As $\vartheta * (\vartheta * \varrho) \leq \varrho$, so by hypothesis

$$\widetilde{\Psi^P}(\vartheta) \geq \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\Psi^P}(\varrho) \text{ and } \widetilde{\Phi^P}(\vartheta) \leq \widetilde{\Phi^P}(\vartheta * \varrho) \vee \widetilde{\Phi^P}(\varrho).$$

Hence $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an *mpCI* of $\widetilde{\mathcal{A}}$. □

Theorem 4.5. *An mpCS $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an mpCI of $\widetilde{\mathcal{A}} \Leftrightarrow$ each non-empty level subset $U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$ is an ideal of $\widetilde{\mathcal{A}}$, $\forall\ \widehat{[\Theta, \theta]} = \big([\Theta_1, \theta_1], [\Theta_2, \theta_2], \ldots, [\Theta_m, \theta_m]\big) \in D[0,1]^m$ and $\hat{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) \in [0,1]^m$.*

Proof. ($\Rightarrow$) Suppose that $\vartheta * \varrho, \varrho \in U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$. Then $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \geq [\Theta_\imath, \theta_\imath]$, $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \leq \varepsilon_\imath$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho) \geq [\Theta_\imath, \theta_\imath]$, $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho) \leq \varepsilon_\imath\ \forall\ 1 \leq \imath \leq m$. Since $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an *mpCI* of $\widetilde{\mathcal{A}}$, so we have

$$\begin{aligned}
\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho) \\
&\geq [\Theta_\imath, \theta_\imath] \wedge [\Theta_\imath, \theta_\imath] \\
&= [\Theta_\imath, \theta_\imath]
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho) \\
&\leq \varepsilon_\imath \vee \varepsilon_\imath \\
&= \varepsilon_\imath.
\end{aligned}$$

Therefore $\vartheta \in U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$.

($\Leftarrow$) Suppose that $U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$ is ideal of $\widetilde{\mathcal{A}}$, $\forall\ \widehat{[\Theta, \theta]} = \big([\Theta_1, \theta_1], [\Theta_2, \theta_2], \ldots, [\Theta_m, \theta_m]\big) \in D[0,1]^m$ and $\widehat{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) \in [0,1]^m$. If $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(0) < \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta)$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(0) > \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta)$ for some $\vartheta, \varrho \in \widetilde{\mathcal{A}}$. Choose $\widehat{[\delta, \widetilde{\Phi^P}]} = ([\delta_1, \gamma_1], [\delta_2, \gamma_2], ..., [\delta_m, \gamma_m]) \in D[0,1]^m$ and $\hat{\ell} = (\ell_1, \ell_2, ..., \ell_m) \in [0,1]^m$ such that $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(0) < [\delta_\imath, \gamma_\imath] \leq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta)$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(0) > \ell_\imath \geq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta)$ for each $1 \leq \imath \leq m$ implies $\vartheta \in U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$ but $0 \notin U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$, a contradiction. So $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(0) \geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta)$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(0) \leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta)\ \forall\ \vartheta \in \widetilde{\mathcal{A}}$ and $1 \leq \imath \leq m$. Again, if $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) < \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\}$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) > \widetilde{\varpi_\imath} \vee \circ \widetilde{\Phi^P}(\vartheta * \varrho) \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho)$ for some $\vartheta, \varrho \in \widetilde{\mathcal{A}}$. Choose $\widehat{[\delta, \widetilde{\Phi^P}]} = ([\delta_1, \gamma_1], [\delta_2, \gamma_2], ..., [\delta_m, \gamma_m]) \in D[0,1]^m$ and $\hat{\ell} = (\ell_1, \ell_2, ..., \ell_m) \in [0,1]^m$ such that $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) < [\delta_\imath, \gamma_\imath] \leq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) > \ell_\imath \geq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho)$ for each $1 \leq \imath \leq m$ implies $\vartheta * \varrho \in U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$ and $\varrho \in U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$ but $\vartheta \notin U(\widetilde{\Psi^P}; \widehat{[\Theta, \theta]}, \hat{\varepsilon})$, which is a contradiction. So $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) \geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)$ and $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) \leq$

$\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho)\ \forall\, \vartheta, \varrho \in S$ and $1 \leq \imath \leq m$. Hence $\mathcal{C}_\mathcal{S} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an *mpCI* of $\widetilde{\mathcal{A}}$ □

5. m-POLAR CUBIC COMMUTATIVE IDEALS

The notion of *mpC* commutative ideal of *BCK/BCI*-algebras is defined in this section. Some connections between *mpC* subalgebras, *mpC* ideals, and *mpC* commutative ideals are studied.

Definition 5.1. An *mpCS* $\mathcal{C}_\mathcal{S} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ of $\widetilde{\mathcal{A}}$ is called an *mpC* commutative ideal (briefly, *mpCCI*) if it satisfies (C3) and the following conditions:

(C6) $(\,\forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}})\ \widetilde{\Psi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \widetilde{\Psi^P}((\vartheta * \varrho) * \hbar) \wedge \widetilde{\Psi^P}(\hbar)$,

(C7) $(\,\forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}})\ \widetilde{\Phi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \leq \widetilde{\Phi^P}((\vartheta * \varrho) * \hbar) \vee \widetilde{\Phi^P}(\hbar)$,

that is,

(C6) $\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}((\vartheta * \varrho) * \hbar) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\hbar)$,

(C7) $\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}((\vartheta * \varrho) * \hbar) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\hbar)$,

$\forall\, \vartheta, \varrho, \hbar \in \widetilde{\mathcal{A}}$ and $1 \leq \imath \leq m$.

Example 5.2. Consider a BCK-algebra $\widetilde{\mathcal{A}}$ of Example 3.3. Let $[\omega_j, \varphi_j], [\Theta_j, \theta_j], [\delta_j, \gamma_j] \in D[0,1]^m$ be such that $[\omega_j, \varphi_j] \geq [\Theta_j, \theta_j] \geq [\delta_j, \gamma_j]\ \forall\ j \in \{1, 2, ..., m\}$. Now define an *mpCS* $\mathcal{C}_\mathcal{S} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ on $\widetilde{\mathcal{A}}$ as:

$*$	$\widetilde{\Psi^P}$	$\widetilde{\Phi^P}$
0	$([\omega_1, \varphi_1], [\omega_2, \varphi_2], \ldots, [\omega_m, \varphi_m])$	$(0.3, 0.3, ..., 0.3)$
ϑ	$([\Theta_1, \theta_1], [\Theta_2, \theta_2], \ldots, [\Theta_m, \theta_m])$	$(0.3, 0.3, ..., 0.3)$
ϱ	$([\delta_1, \gamma_1], [\delta_2, \gamma_2], \ldots, [\delta_m, \gamma_m])$	$(0.3, 0.3, ..., 0.3)$
ℓ	$([\delta_1, \gamma_1], [\delta_2, \gamma_2], \ldots, [\delta_m, \gamma_m])$	$(0.3.0.3, ..., 0.3)$

It is straightforward to verify that $\mathcal{C}_\mathcal{S}$ is an $mCCI$ of $\widetilde{\mathcal{A}}$.

Theorem 5.1. *Every mpCCI of BCK-algebra $\widetilde{\mathcal{A}}$ is an mpCI of $\widetilde{\mathcal{A}}$.*

Proof. Let $\mathcal{C}_\mathcal{S} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ be any $mpCCI$ of $\widetilde{\mathcal{A}}$ and $\vartheta, \varrho, \hbar \in \widetilde{\mathcal{A}}$. Then, we have

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta) &= \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * (0 * (0 * \vartheta))) \\ &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}((\vartheta * 0) * \varrho) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\} \\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\varrho)\}.\end{aligned}$$

and

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta) &= \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * (0 * (0 * \vartheta))) \\ &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}((\vartheta * 0) * \varrho) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho) \\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\varrho).\end{aligned}$$

Hence $\mathcal{C}_\mathcal{S} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an *mpCI* of $\widetilde{\mathcal{A}}$. □

Corollary 5.2. *Every mpCCI of $\widetilde{\mathcal{A}}$ is an mpCSub of $\widetilde{\mathcal{A}}$.*

Remark. *In general, converse of Theorem 5.1 does not hold.*

Example 5.3. Consider a BCK-algebra $\widetilde{\mathcal{A}} = \{0, \vartheta, \jmath, \varrho, \ell\}$ with the following table.

$*$	0	ϑ	$\jmath$	ϱ	ℓ
0	0	0	0	0	0
ϑ	ϑ	0	ϑ	0	0
$\jmath$	$\jmath$	$\jmath$	0	0	0
ϱ	ϱ	ϱ	ϱ	0	0
ℓ	ℓ	ℓ	ϱ	$\jmath$	0

TABLE 7. Caley table for $*$-opertaion

Let $\widehat{[\varsigma,\delta]} = ([\varsigma_1,\delta_1],[\varsigma_2,\delta_2],\ldots,[\varsigma_m,\delta_m]), \widehat{[\psi,\phi]} = ([\psi_1,\phi_1],[\psi_2,\phi_2],\ldots,[\psi_m,\phi_m]), \widehat{[\rho,\sigma]} = ([\rho_1,\sigma_1],[\rho_2,\sigma_2],\ldots,[\rho_m,\sigma_m]) \in D[0,1]^m$ and $\widehat{\Theta} = (\Theta_1,\Theta_2,...,\Theta_m), \widehat{\theta} = (\theta_1,\theta_2,...,\theta_m), \widehat{\gamma} = (\gamma_1,\gamma_2,...,\gamma_m) \in [0,1]^m$ be such that $\widehat{[\theta,\delta]} \geq \widehat{[\psi,\phi]} \geq \widehat{[\rho,\sigma]}$ and $\widehat{\Theta} \leq \widehat{\theta} \leq \widehat{\gamma}$. Now define an *mpCS* $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ on $\widetilde{\mathcal{A}}$ as:

$*$	$\widetilde{\Psi^P}$	δ
0	$\widehat{[\varsigma,\delta]} = ([\varsigma_1,\delta_1],[\varsigma_2,\delta_2],\ldots,[\varsigma_m,\delta_m])$	$\widehat{\Theta} = (\Theta_1,\Theta_2,...,\Theta_m)$
ϑ	$\widehat{[\psi,\phi]} = ([\psi_1,\phi_1],[\psi_2,\phi_2],\ldots,[\psi_m,\phi_m])$	$\widehat{\theta} = (\theta_1,\theta_2,...,\theta_m)$
$\jmath$	$\widehat{[\rho,\sigma]} = ([\rho_1,\sigma_1],[\rho_2,\sigma_2],\ldots,[\rho_m,\sigma_m])$	$\widehat{\gamma} = (\gamma_1,\gamma_2,...,\gamma_m)$
ϱ	$\widehat{[\rho,\sigma]} = ([\rho_1,\sigma_1],[\rho_2,\sigma_2],\ldots,[\rho_m,\sigma_m])$	$\widehat{\gamma} = (\gamma_1,\gamma_2,...,\gamma_m)$
ℓ	$\widehat{[\rho,\sigma]} = ([\rho_1,\sigma_1],[\rho_2,\sigma_2],\ldots,[\rho_m,\sigma_m])$	$\widehat{\gamma} = (\gamma_1,\gamma_2,...,\gamma_m)$

TABLE 8. Table for membership values

It is easy to verify that $\widetilde{\Psi^P}$ is an *mpCI* of $\widetilde{\mathcal{A}}$ but not an *mpCCI* of $\widetilde{\mathcal{A}}$ beacuse $[\rho_1,\sigma_1] = (\pi_1 \circ \widetilde{\Psi^P})(\jmath) = (\pi_1 \circ \widetilde{\Psi^P})(\jmath * (\varrho * (\varrho * \jmath))) \ngeq \pi_1 \circ \widetilde{\Psi^P}((\jmath * \varrho) * 0) \wedge \pi_1 \circ \widetilde{\Psi^P}(0) = (\pi_1 \circ \widetilde{\Psi^P})(0) = [\theta_1,\delta_1]$.

Theorem 5.3. *Let $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ be an mpCI of $\widetilde{\mathcal{A}}$. Then $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an mpCCI $\Leftrightarrow \forall\, \vartheta, \varrho \in \widetilde{\mathcal{A}}$,*

$$\widetilde{\Psi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \widetilde{\Psi^P}(\vartheta * \varrho) \text{ and } \widetilde{\Phi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \leq \widetilde{\Phi^P}(\vartheta * \varrho).$$

Proof. ($\Rightarrow$) Let $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ be an *mpCCI* of $\widetilde{\mathcal{A}}$. Then $\forall\, \vartheta, \varrho, \hbar \in \widetilde{\mathcal{A}}$, we have

$$\widehat{\widetilde{\varpi_\imath}} \circ \widetilde{\Psi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}((\vartheta * \varrho) * \hbar) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\hbar)$$

and

$$\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}((\vartheta * \varrho) * \hbar) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\hbar)$$

Taking $\hbar = 0$, so

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * (\varrho * (\varrho * \vartheta)) &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}((\vartheta * \varrho) * 0) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(0)\\ &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho)\\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * (\varrho * (\varrho * \vartheta)) &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}((\vartheta * \varrho) * 0) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(0)\\ &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho) \vee \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho)\\ &= \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho).\end{aligned}$$

($\Leftarrow$) Let $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ be an *mpCI* such that $\forall\ \vartheta, \varrho, \hbar \in \widetilde{\mathcal{A}}$ with $\vartheta * \varrho \leq \hbar$ implies $\widetilde{\Psi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \widetilde{\Psi^P}(\vartheta * \varrho)$ and $\widetilde{\Phi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \widetilde{\Phi^P}(\vartheta * \varrho)$. By assumption, we have

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\vartheta * \varrho)\\ &\geq \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}((\vartheta * \varrho) * \hbar) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Psi^P}(\hbar)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\vartheta * \varrho)\\ &\leq \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}((\vartheta * \varrho) * \hbar) \wedge \widetilde{\varpi_\imath} \circ \widetilde{\Phi^P}(\hbar).\end{aligned}$$

Therefore $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an $mpCCI$ of $\widetilde{\mathcal{A}}$. □

Theorem 5.4. *Every mpCI of commutative BCK-algebra $\widetilde{\mathcal{A}}$ is an $mpCCI$.*

Proof. Let $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ be an *mpCI* of $\widetilde{\mathcal{A}}$. Then $\forall\ \vartheta, \varrho, \hbar \in \widetilde{\mathcal{A}}$, we have

$$\begin{aligned}\Big(\big(\vartheta * (\varrho * (\varrho * \vartheta))\big) * \big((\vartheta * \varrho) * \hbar\big)\Big) * \hbar &= \Big(\big(\vartheta * (\varrho * (\varrho * \vartheta))\big) * \hbar\Big) * \big((\vartheta * \varrho) * \hbar\big)\\ &\leq \big(\vartheta * (\varrho * (\varrho * \vartheta))\big) * (\vartheta * \varrho)\\ &= \big(\vartheta * (\vartheta * \varrho))\big) * \big(\varrho * (\varrho * \vartheta))\big)\\ &= 0\end{aligned}$$

It follows that $\Big(\big(\vartheta * (\varrho * (\varrho * \vartheta))\big) * \big((\vartheta * \varrho) * \hbar\big)\Big) \leq \hbar$. As $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an *mpCI* of $\widetilde{\mathcal{A}}$, so by Lemma 4.2, $\widetilde{\Psi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \geq \widetilde{\Psi^P}((\vartheta * \varrho) * \hbar) \wedge \widetilde{\Psi^P}(\hbar)$ and $\widetilde{\Phi^P}(\vartheta * (\varrho * (\varrho * \vartheta))) \leq \widetilde{\Phi^P}((\vartheta * \varrho) * \hbar) \vee \widetilde{\Phi^P}(\hbar)$. Hence $\mathcal{C}_{\mathcal{S}} = (\widetilde{\Psi^P}, \widetilde{\Phi^P})$ is an $mpCCI$ of $\widetilde{\mathcal{A}}$. □

6. CONCLUSION

The *mpCS* provides a new structure with more precision, flexibility and compatibility when more than one variable needs to be taken. The notion of *mpCS* s is therefore much wider than the notion of cubic sets. It is therefore necessary to apply the *mpCS* s to applications. We constructed the ideal theory in BCK/BCI-algebras based on mpC structures in this study. We developed the ideas of mpC subalgebras, mpC ideals, and mpC commutative ideals. We showed that mpC ideals are mpC subalgebra, but the converse is not true, as illustrated by an example in this support. We defined a condition under which an mpC subalgebra transforms into an mpC ideals. In addition, we defined mpC ideals in terms of fuzzy ideals and *BCK/BCI*-algebra ideals. Furthermore, we established that mpC commutative ideals are mpC fuzzy ideals, but the converse is not true, as illustrated by an example . A condition under which the mpC ideal becomes a mpC commutative ideal is also presented.

The work offers a new platform in this field for future research and related fields. In fact, this study will serve as a basis for further analysis of the *m*-polar cubic structures in related algebraic structures. The notion presented in this work can be further extended to various algebras such as *R0*-algebras, *BL*-algebras ,*MTL*-algebras, *UP*-algebras, *MV*-algebras, *EQ*-algebras and lattice implication algebras, etc.

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