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BEHAVIOR AND FORMULA OF THE SOLUTIONS OF RATIONAL DIFFERENCE EQUATIONS OF ORDER SIX

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ABSTRACT. This paper is devoted to find the form of the solution of the following rational difference equations :

$$x_{n+1} = \frac{x_{n-3}x_{n-5}}{x_{n-1}(\pm 1 \pm x_{n-3}x_{n-5})},$$

where the initial conditions $x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary non zero real numbers. Also, we study the behavior of the solutions.

1. INTRODUCTION

In this paper, we obtain the solution of the following difference equations :

$$x_{n+1} = \frac{x_{n-3}x_{n-5}}{x_{n-1}(\pm 1 \pm x_{n-3}x_{n-5})}, \tag{1.1}$$

where the inital conditions $x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary non zero real numbers. Also, we study the solution of some special equations. Many researchers have investigated the behavior of the solution of rational difference equations for instance:

Cinar [4] discussed the solutions of the following difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

Ibrahim [16] gave the solutions of the following difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}.$$

Karatas et al [17] supplied the solution to the difference equation below

$$x_{n+1} = \frac{x_{n-5}}{(1 + x_{n-2}x_{n-5})}.$$

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Zayed [26] discussed the dynamics of the difference equation

$$x_{n+1} = Ax_n + B_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}.$$

Saleh [24] analyzed the stability and periodicity of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}.$$

Elsayed [5] discussed the stability of the rational difference equation

$$x_{n+1} = \frac{a + bx_{n-1} + cx_{n-k}}{dx_{n-1} + ex_{n-k}}.$$

For some results about difference equations can be see the references [1-27].

Definition: let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function.Then, for every set of initial condition $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}, ..., x_{n-k}), n = 0, 1, ..., \tag{1.2}$$

has a unique solation $\{x_n\}_{n=-k}^{\infty}$ [15].

The linearized equation of Eq.(1.2) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, ..., \bar{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem A [15]: Assume that $p_i \in R, i = 1, 2, ..., k$ and $k \in 0, 1, 2, ...$. Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, n = 0, 1, ...$$

2. The First Equation $x_{n+1} = \frac{x_{n-3}x_{n-5}}{x_{n-1}(1+x_{n-3}x_{n-5})}$

In this section, we give a specific form of the solution of the first difference equation in the form

$$x_{n+1} = \frac{x_{n-3}x_{n-5}}{x_{n-1}(1 + x_{n-3}x_{n-5})}, \tag{2.1}$$

where the initial values are arbitrary non zero real numbers.

Theorem 2.1. Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq. (2.1). Then for $n = 0, 1, ...,$

$$x_{4n-3} = \frac{df^n}{b^n} \prod_{i=0}^{n-1} \frac{(1 + ibd)}{(1 + (i+1)df)},$$

$$x_{4n-2} = \frac{ce^n}{a^n} \prod_{i=0}^{n-1} \frac{(1+iac)}{(1+(i+1)ce)},$$

$$x_{4n-1} = \frac{b^{n+1}}{f^n} \prod_{i=0}^{n-1} \frac{(1+(i+1)df)}{(1+(i+1)bd)},$$

$$x_{4n} = \frac{a^{n+1}}{e^n} \prod_{i=0}^{n-1} \frac{(1+(i+1)ce)}{(1+(i+1)ac)},$$

where $x_{-5} = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

Proof: For n=0, the result holds. Now, suppose that $n > 0$ and that our assumption holds for $n-1$. That is

$$x_{4n-7} = \frac{df^{n-1}}{b^{n-1}} \prod_{i=0}^{n-2} \frac{(1+ibd)}{(1+(i+1)df)},$$

$$x_{4n-6} = \frac{ce^{n-1}}{a^{n-1}} \prod_{i=0}^{n-2} \frac{(1+iac)}{(1+(i+1)ce)},$$

$$x_{4n-5} = \frac{b^n}{f^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(i+1)df)}{(1+(i+1)bd)},$$

$$x_{4n-4} = \frac{a^n}{e^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(i+1)ce)}{(1+(i+1)ac)}.$$

Now, it follows from Eq. (2.1) that

$$x_{4n} = \frac{x_{4n-4}x_{4n-6}}{x_{4n-2}(1+x_{4n-4}x_{4n-6})}$$

$$= \frac{\frac{a^n}{e^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(i+1)ce)}{(1+(i+1)ac)} \frac{ce^{n-1}}{a^{n-1}} \prod_{i=0}^{n-2} \frac{(1+iac)}{(1+(i+1)ce)}}{\frac{ce^n}{a^n} \prod_{i=0}^{n-1} \frac{(1+iac)}{(1+(i+1)ce)} (1 + \frac{a^n}{e^{n-1}} \prod_{i=0}^{n-2} \frac{(1+(i+1)ce)}{(1+(i+1)ac)} \frac{ce^{n-1}}{a^{n-1}} \prod_{i=0}^{n-2} \frac{(1+iac)}{(1+(i+1)ce)})}$$

$$= \frac{ac \prod_{i=0}^{n-2} \frac{(1+iac)}{(1+(i+1)ac)}}{\frac{ce^n}{a^n} \prod_{i=0}^{n-1} \frac{(1+iac)}{(1+(i+1)ce)} (1 + ac \prod_{i=0}^{n-2} \frac{(1+iac)}{(1+(i+1)ac)})}$$

$$= \frac{a^{n+1}c}{ce^n(1+nac) \prod_{i=0}^{n-1} \frac{(1+iac)}{(1+(i+1)ce)} \frac{(1+nac+ac)}{(1+nac)}}$$

$$= \frac{a^{n+1}}{e^n} \prod_{i=0}^{n-1} \frac{(1+(i+1)ce)}{(1+iac)} \frac{1}{(1+(n+1)ac)}.$$

Hence,we have

$$x_{4n} = \frac{a^{n+1}}{e^n} \prod_{i=0}^{n-1} \frac{(1+(i+1)ce)}{(1+(i+1)ac)}.$$

Similarly, we see that

$$x_{4n-1} = \frac{x_{4n-5}x_{4n-7}}{x_{4n-3}(1+x_{4n-5}x_{4n-7})}$$

$$= \frac{\frac{b^n}{f^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(i+1)df)}{(1+(i+1)bd)}\frac{df^{n-1}}{b^{n-1}}\prod_{i=0}^{n-2}\frac{(1+ibd)}{(1+(i+1)df)}}{\frac{df^n}{b^n}\prod_{i=0}^{n-1}\frac{(1+ibd)}{(1+(i+1)df)}(1+\frac{b^n}{f^{n-1}}\prod_{i=0}^{n-2}\frac{(1+(i+1)df)}{(1+(i+1)bd)}\frac{df^{n-1}}{b^{n-1}}\prod_{i=0}^{n-2}\frac{(1+ibd)}{(1+(i+1)df)})}$$

$$= \frac{bd\prod_{i=0}^{n-2}\frac{(1+ibd)}{(1+(i+1)bd)}}{\frac{df^n}{b^n}\prod_{i=0}^{n-1}\frac{(1+ibd)}{(1+(i+1)df)}(1+bd\prod_{i=0}^{n-2}\frac{(1+ibd)}{(1+(i+1)bd)})}$$

$$= \frac{b^{n+1}d}{df^n(1+nbd)\prod_{i=0}^{n-1}\frac{(1+ibd)}{(1+(i+1)df)}\frac{(1+nbd+bd)}{(1+nbd)}}$$

$$= \frac{b^{n+1}}{f^n}\prod_{i=0}^{n-1}\frac{(1+(i+1)df)}{(1+ibd)}\frac{1}{(1+(n+1)bd)}.$$

Hence,we have

$$x_{4n-1} = \frac{b^{n+1}}{f^n}\prod_{i=0}^{n-1}\frac{(1+(i+1)df)}{(1+(i+1)bd)}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 2.2. Eq. (2.1) has a unique equilibrium point which is $\bar{x} = 0$, and is not locally asympotoically stable.

Proof: From Eq. (2.1), we see that

$$\bar{x} = \frac{\bar{x}^2}{\bar{x}(1+\bar{x}^2)},$$

or

$$\bar{x}^2(1+\bar{x}^2-1) = 0, \Rightarrow \bar{x}^4 = 0.$$

Thus the equilibrium point of Eq. (2.1) is $\bar{x} = 0$.

Let $f : (0,\infty)^3 \to (0,\infty)$ be a continuously differentiable function defined by

$$f(u,v,w) = \frac{vw}{u(1+vw)}.$$

Therefore it follows that

$$f_u(u,v,w) = \frac{-vw}{u^2(1+vw)},\ f_v(u,v,w) = \frac{w}{u(1+vw)^2},\ f_w(u,v,w) = \frac{v}{u(1+vw)^2},$$

we obtain $f_u(\bar{x},\bar{x},\bar{x}) = -1,\ f_v(\bar{x},\bar{x},\bar{x}) = 1,\ f_w(\bar{x},\bar{x},\bar{x}) = 1.$

The proof follows by using Theorem A.

Example 2.1. This Fig.1 Shwo the solution when $x_{-5}=2,\ x_{-4}=20,\ x_{-3}=4,\ x_{-2}=-3,\ x_{-1}=1,\ x_0=1$.
Example 2.2. See Fig.2 where we put the initials $x_{-5}=1,\ x_{-4}=3,\ x_{-3}=5,\ x_{-2}=11,\ x_{-1}=0.5,\ x_0=-1$.

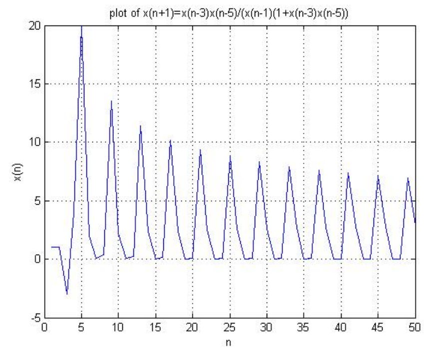

FIGURE 1. The behavior of the solution of Eq. (2.1).

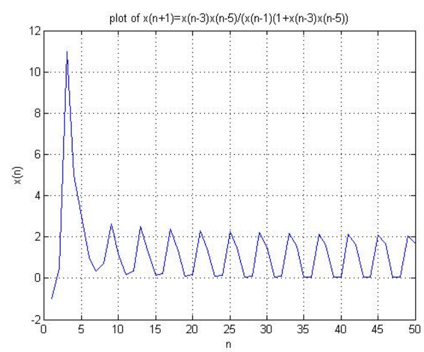

FIGURE 2. The stability of the solution of Eq. (2.1).

3. THE SECOND EQUATION $x_{n+1}=\frac{x_{n-3}x_{n-5}}{x_{n-1}(-1+x_{n-3}x_{n-5})}$

In this section is devoted to obtain the solution of the second difference equation which is

$$x_{n+1}=\frac{x_{n-3}x_{n-5}}{x_{n-1}(-1+x_{n-3}x_{n-5})}, \tag{3.1}$$

where $x_{-3}x_{-5},\ x_{-2}x_{-4},\ x_{-1}x_{-3},\ x_0x_{-2}\neq 1$.

Theorem 3.1. Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq. (3.1). Then for $n=0,1,...,$

$$x_{8n-5}=\frac{b^{2n}(-1+df)^n}{f^{2n-1}(-1+bd)^n},$$

$$x_{8n-4}=\frac{a^{2n}(-1+ce)^n}{e^{2n-1}(-1+ac)^n},$$

$$x_{8n-3} = \frac{df^{2n}(-1+bd)^n}{b^{2n}(-1+df)^n},$$
$$x_{8n-2} = \frac{ce^{2n}(-1+ac)^n}{a^{2n}(-1+ce)^n},$$
$$x_{8n-1} = \frac{b^{2n+1}(-1+df)^n}{f^{2n}(-1+bd)^n},$$
$$x_{8n} = \frac{a^{2n+1}(-1+ce)^n}{e^{2n}(-1+ac)^n},$$
$$x_{8n+1} = \frac{df^{2n+1}(-1+bd)^n}{b^{2n+1}(-1+df)^{n+1}},$$
$$x_{8n+2} = \frac{ce^{2n+1}(-1+ac)^n}{a^{2n+1}(-1+ce)^{n+1}}.$$

Proof: For n=0, the result holds. Now, suppose that $n > 0$ and that our assumption holds for $n-1$. That is

$$x_{8n-13} = \frac{b^{2n-2}(-1+df)^{n-1}}{f^{2n-3}(-1+bd)^{n-1}},$$
$$x_{8n-12} = \frac{a^{2n-2}(-1+ce)^{n-1}}{e^{2n-3}(-1+ac)^{n-1}},$$
$$x_{8n-11} = \frac{df^{2n-2}(-1+bd)^{n-1}}{b^{2n-2}(-1+df)^{n-1}},$$
$$x_{8n-10} = \frac{ce^{2n-2}(-1+ac)^{n-1}}{a^{2n-2}(-1+ce)^{n-1}},$$
$$x_{8n-9} = \frac{b^{2n-1}(-1+df)^{n-1}}{f^{2n-2}(-1+bd)^{n-1}},$$
$$x_{8n-8} = \frac{b^{2n-1}(-1+ce)^{n-1}}{e^{2n-2}(-1+ac)^{n-1}},$$
$$x_{8n-7} = \frac{df^{2n-1}(-1+bd)^{n-1}}{b^{2n-1}(-1+df)^{n}},$$
$$x_{8n-6} = \frac{ce^{2n-1}(-1+ac)^{n-1}}{a^{2n-1}(-1+ce)^{n}}.$$

it follows from Eq. (3.1) that

$$x_{8n-1} = \frac{x_{8n-5}x_{8n-7}}{x_{8n-3}(-1+x_{8n-5}x_{8n-7})}$$

$$= \frac{\frac{b^{2n}(-1+df)^n}{f^{2n-1}(-1+bd)^n}\frac{df^{2n-1}(-1+bd)^{n-1}}{b^{2n-1}(-1+df)^n}}{\frac{df^{2n}(-1+bd)^n}{b^{2n}(-1+df)^n}(-1+\frac{b^{2n}(-1+df)^n}{f^{2n-1}(-1+bd)^n}\frac{df^{2n-1}(-1+bd)^{n-1}}{b^{2n-1}(-1+df)^n})}.$$

Hence, we have

$$x_{8n-1} = \frac{b^{2n+1}(-1+df)^n}{f^{2n}(-1+bd)^n}.$$

Similarly, we see that

$$x_{8n-2} = \frac{x_{8n-6}x_{8n-8}}{x_{8n-4}(-1+x_{8n-6}x_{8n-8})}$$

$$= \frac{\frac{ce^{2n-1}(-1+ac)^{n-1}}{a^{2n-1}(-1+ce)^{n}}\frac{a^{2n-1}(-1+ce)^{n-1}}{e^{2n-2}(-1+ac)^{n-1}}}{\frac{a^{2n}(-1+ce)^{n}}{e^{2n-1}(-1+ac)^{n}}(-1+\frac{ce^{2n-1}(-1+ac)^{n-1}}{a^{2n-1}(-1+ce)^{n}}\frac{a^{2n-1}(-1+ce)^{n-1}}{e^{2n-2}(-1+ac)^{n-1}})}.$$

Then

$$x_{8n-2} = \frac{ce^{2n}(-1+ac)^{n}}{a^{2n}(-1+ce)^{n}}.$$

Similarly, one can simply prove the other relations. Thus, the proof is completed.

Theorem 3.2. Eq. (3.1) has three equilibrium point which are $0, \pm\sqrt{2}$, and are not locally asympotoically stable.

Proof: From Eq. (3.1), we see that

$$\bar{x} = \frac{\bar{x}^2}{\bar{x}(-1+\bar{x}^2)}.$$

Then

$$\bar{x}^2(\bar{x}^2-2) = 0.$$

Thus the equilibrium point of Eq. (3.1) are $\bar{x} = 0, \pm\sqrt{2}$.

Let $f : (0,\infty)^3 \to (0,\infty)$ be a continuously differentiable function defined by

$$f(u,v,w) = \frac{vw}{u(-1+vw)}.$$

Therefore it follows that

$$f_u(u,v,w) = \frac{-vw}{u^2(-1+vw)},$$
$$f_v(u,v,w) = \frac{-w}{u(-1+vw)^2},$$
$$f_w(u,v,w) = \frac{-v}{u(-1+vw)^2},$$

we see that $f_u(\bar{x},\bar{x},\bar{x}) = \pm 1,\ f_v(\bar{x},\bar{x},\bar{x}) = -1,\ f_w(\bar{x},\bar{x},\bar{x}) = -1.$

The proof follows by using Theorem A.

Example 3.1. We assume $x_{-5} = -5,\ x_{-4} = 3,\ x_{-3} = 1,\ x_{-2} = 0.1,\ x_{-1} = 15,\ x_0 = 1$. See Fig.3.

Example 3.2. See Fig.4 when we take the initials $x_{-5} = 10, x_{-4} = 5,\ x_{-3} = 2,\ x_{-2} = -1,\ x_{-1} = 4,\ x_0 = 8$.

Lemma 3.1. Eq. (3.1) has a periodic solutions of period four iff $x_{-3}x_{-5} = x_{-2}x_{-4} = x_{-1}x_{-3} = x_0x_{-2} = 2$ and $x_{-1} = x_{-5},\ x_0 = x_{-4}$, and will be take the form $\{x_{-1}, x_0, x_{-3}, x_{-2}, ...\}$.

Proof: Suppose that there exists a prime period four solution of Eq. (3.1) of the form

$$x_{-1}, x_0, x_{-3}, x_{-2}, x_{-1}, x_0, x_{-3}, x_{-2}, ...$$

Then we see from the form of solution of Eq. (3.1) that

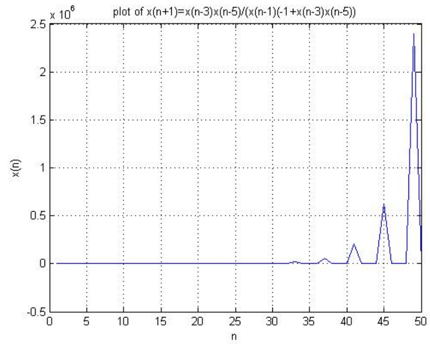

FIGURE 3. The behaviour of the solution of Eq. (3.1).

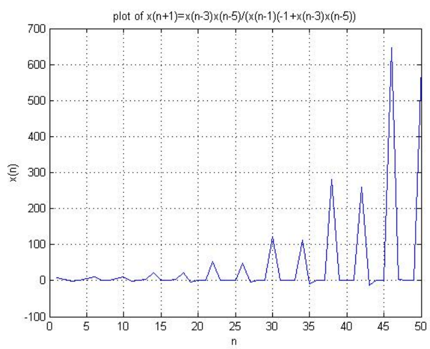

FIGURE 4. The stability of the solution of Eq. (3.1).

$$x_{8n-5}=\frac{b^{2n}}{f^{2n-1}},\ x_{8n-4}=\frac{a^{2n}}{e^{2n-1}},\ x_{8n-3}=\frac{df^{2n}}{b^{2n}},\ x_{8n-2}=\frac{ce^{2n}}{a^{2n}},$$
$$x_{8n-1}=\frac{b^{2n+1}}{f^{2n}},\ x_{8n}=\frac{a^{2n+1}}{e^{2n}},\ x_{8n+1}=\frac{df^{2n+1}}{b^{2n+1}},\ x_{8n+2}=\frac{ce^{2n+1}}{a^{2n+1}}.$$

Then

$$b=f,\ a=e.$$

Hence, we have

$$x_{8n-5}=b,\ x_{8n-4}=a,\ x_{8n-3}=d,\ x_{8n-2}=c,$$
$$\mathrm{x}_{8n-1}=b,\ x_{8n}=a,\ x_{8n+1}=d,\ x_{8n+2}=c.$$

Thus we have a period four solution and the proof is complete.
For confirming the result of this lemma, we consider numerical example for $x_{-5}=5,\ x_{-4}=\frac{2}{5},\ x_{-3}=\frac{2}{5},\ x_{-2}=5,\ x_{-1}=5,\ x_0=\frac{2}{5}$. See Fig.5.

Lemma 3.2. Eq. (3.1) has a periodic solutions of period eight iff $x_{-1}=x_{-5},\ x_0=x_{-4}$, and will be take the form $\{x_{-1},x_0,x_{-3},x_{-2},x_{-1},x_0,\frac{x_{-3}}{(-1+x_{-1}x_{-3})},\frac{x_{-2}}{(-1+x_{-2}x_{-4})},...\}$.
Proof: Suppose that there exists a prime period eight solution of Eq. (3.1) of the form

$$x_{-1},x_0,x_{-3},x_{-2},x_{-1},x_0,\frac{x_{-3}}{(-1+x_{-1}x_{-3})},\frac{x_{-2}}{(-1+x_{-2}x_{-4})},$$

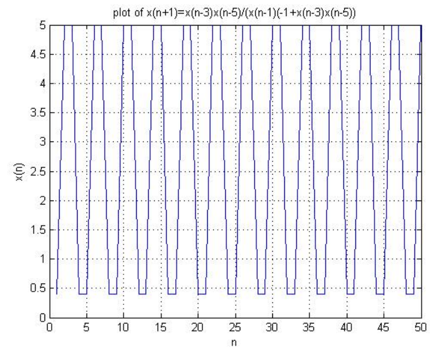

FIGURE 5. Eq. (3.1) has period four solutions.

$$x_{-1}, x_0, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{x_{-3}}{(-1+x_{-1}x_{-3})}, \frac{x_{-2}}{(-1+x_{-2}x_{-4})}, \dots .$$

Then we see from the form of solution of Eq. (3.1) that

$$x_{8n-5} = \frac{b^{2n}(-1+df)^n}{f^{2n-1}(-1+bd)^n} = b,\ x_{8n-4} = \frac{a^{2n}(-1+ce)^n}{e^{2n-1}(-1+ac)^n} = a,$$
$$x_{8n-3} = \frac{df^{2n}(-1+bd)^n}{b^{2n}(-1+df)^n} = d,\ x_{8n-2} = \frac{ce^{2n}(-1+ac)^n}{a^{2n}(-1+ce)^n} = c,$$
$$x_{8n-1} = \frac{b^{2n+1}(-1+df)^n}{f^{2n}(-1+bd)^n} = b,\ x_{8n} = \frac{a^{2n+1}(-1+ce)^n}{e^{2n}(-1+ac)^n} = a,$$
$$x_{8n+1} = \frac{df^{2n+1}(-1+bd)^n}{b^{2n+1}(-1+df)^{n+1}} = \frac{d}{(-1+bd)},\ x_{8n+2} = \frac{ce^{2n+1}(-1+ac)^n}{a^{2n+1}(-1+ce)^{n+1}} = \frac{c}{(-1+ce)}.$$

Thus we have a period eight solution and the proof is complete.
Now, we take a numerical example for proving the result of this lemma. We assume $x_{-5} = 5,\ x_{-4} = 1,\ x_{-3} = 3,\ x_{-2} = 0.1,\ x_{-1} = 5,\ x_0 = 1$. See Fig.6.

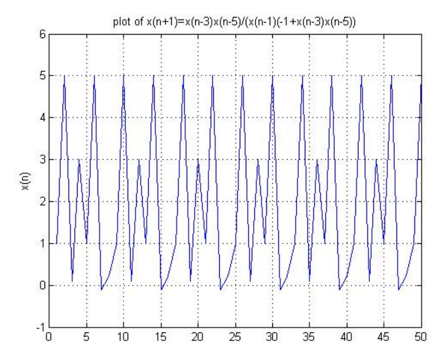

FIGURE 6. Eq. (3.1) has period eight solutions.

The following sections proofs of the theorems and lemmas are similar to those required in the previous sections, thus they will be omitted.

4. THE THIRD EQUATION $x_{n+1}=\frac{x_{n-3}x_{n-5}}{x_{n-1}(1-x_{n-3}x_{n-5})}$

In this section, we will obtain form of the solution of the third difference equation is which is

$$x_{n+1}=\frac{x_{n-3}x_{n-5}}{x_{n-1}(1-x_{n-3}x_{n-5})}. \tag{4.1}$$

Theorem 4.1. Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq. (4.1). Then for $n=0,1,...,$

$$x_{4n-3}=\frac{df^n}{b^n}\prod_{i=0}^{n-1}\frac{(1-ibd)}{(1-(i+1)df)},\ x_{4n-2}=\frac{ce^n}{a^n}\prod_{i=0}^{n-1}\frac{(1-iac)}{(1-(i+1)ce)},$$

$$x_{4n-1}=\frac{b^{n+1}}{f^n}\prod_{i=0}^{n-1}\frac{(1-(i+1)df)}{(1-(i+1)bd)},\ x_{4n}=\frac{a^{n+1}}{e^n}\prod_{i=0}^{n-1}\frac{(1-(i+1)ce)}{(1-(i+1)ac)}.$$

Theorem 4.2. Eq. (4.1) has a unique equilibrium point which is $\bar{x}=0$, and is not locally asympotoically stable.

Example 4.1. We consider $x_{-5}=19,\ x_{-4}=9,\ x_{-3}=2,\ x_{-2}=-3,\ x_{-1}=0.7,$ $x_0=9$. See Fig.7.

Example 4.2. See Fig.8 when we take the initials $x_{-5}=3,\ x_{-4}=5,\ x_{-3}=-1,$ $x_{-2}=9,\ x_{-1}=2,\ x_0=11.$

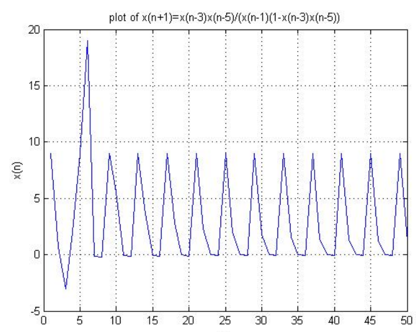

FIGURE 7. Draw the numerical solution of Eq. (4.1).

5. THE FOURTH EQUATION $x_{n+1}=\frac{x_{n-3}x_{n-5}}{x_{n-1}(-1-x_{n-3}x_{n-5})}$

Now, we get the solution form of the fourth difference equation as follows

$$x_{n+1}=\frac{x_{n-3}x_{n-5}}{x_{n-1}(-1-x_{n-3}x_{n-5})}, \tag{5.1}$$

where $x_{-3}x_{-5},\ x_{-2}x_{-4},\ x_{-1}x_{-3},\ x_0x_{-2}\neq -1.$

Theorem 5.1. Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq. (5.1). Then for $n=0,1,...,$

$$x_{8n-5}=\frac{b^{2n}(-1-df)^n}{f^{2n-1}(-1-bd)^n},\ x_{8n-4}=\frac{a^{2n}(-1-ce)^n}{e^{2n-1}(-1-ac)^n},$$

$$x_{8n-3}=\frac{df^{2n}(-1-bd)^n}{b^{2n}(-1-df)^n},\ x_{8n-2}=\frac{ce^{2n}(-1-ac)^n}{a^{2n}(-1-ce)^n},$$

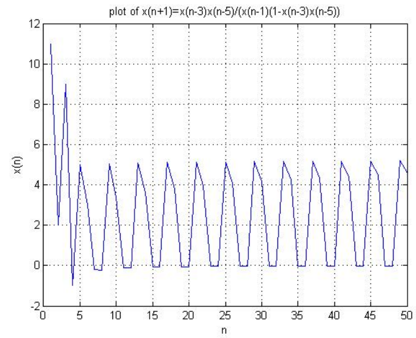

FIGURE 8. The stability of the solution of Eq. (4.1).

$$x_{8n-1} = \frac{b^{2n+1}(-1-df)^n}{f^{2n}(-1-bd)^n},\ x_{8n} = \frac{a^{2n+1}(-1-ce)^n}{e^{2n}(-1-ac)^n},$$
$$x_{8n+1} = \frac{df^{2n+1}(-1-bd)^n}{b^{2n+1}(-1-df)^{n+1}},\ x_{8n+2} = \frac{ce^{2n+1}(-1-ac)^n}{a^{2n+1}(-1-ce)^{n+1}}.$$

Theorem 5.2. Eq. (5.1) has a unique equilibrium point which is $\bar{x} = 0$, and is not locally asympotoically stable.

Example 5.1. Suppose that $x_{-5} = 9,\ x_{-4} = 2,\ x_{-3} = -1,\ x_{-2} = 3,\ x_{-1} = 13,$ $x_0 = 1$. See Fig.9.

Example 5.2. See Fig.10 when we take $x_{-5} = 7,\ x_{-4} = 5,\ x_{-3} = -2,\ x_{-2} = 3,$ $x_{-1} = 9,\ x_0 = 4$.

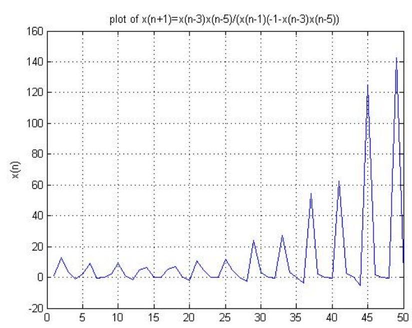

FIGURE 9. The stability of the solution of Eq. (5.1).

Lemma 5.1. Eq. (5.1) has a periodic solutions of period four iff $x_{-3}x_{-5} = x_{-2}x_{-4} = x_{-1}x_{-3} = x_0x_{-2} = -2$ and $x_{-1} = x_{-5},\ x_0 = x_{-4}$, and will be take the form $\{x_{-1}, x_0, x_{-3}, x_{-2}, ...\}$.

We give a numerical example for verifying the result of this lemma. Suppose that $x_{-5} = 8,\ x_{-4} = \frac{-2}{8},\ x_{-3} = \frac{-2}{8},\ x_{-2} = 8,\ x_{-1} = 8,\ x_0 = \frac{-2}{8}$. See Fig.11.

Lemma 5.2. Eq. (5.1) has a periodic solutions of period eight iff $x_{-1} = x_{-5},\ x_0 = x_{-4}$, and will be take the form $\{x_{-1}, x_0, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{x_{-3}}{(-1-x_{-1}x_{-3})}, \frac{x_{-2}}{(-1-x_{-2}x_{-4})}, ...\}$.

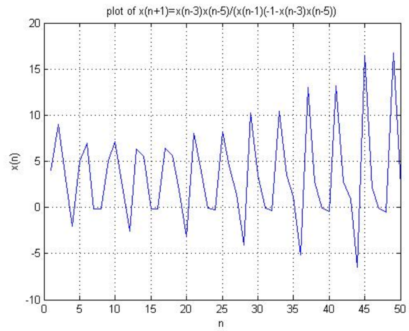

FIGURE 10. The numerical solution of Eq. (5.1).

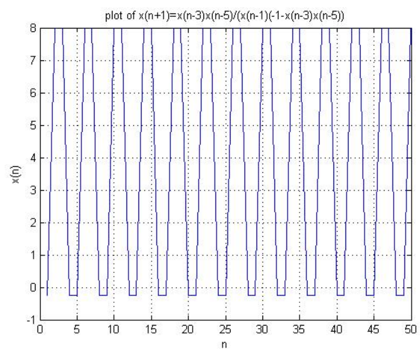

FIGURE 11. Eq. (5.1) has period four solutions.

Now, we take a numerical example for proving the result of this lemma. We assume $x_{-5} = 1,\ x_{-4} = 4,\ x_{-3} = 2,\ x_{-2} = 8,\ x_{-1} = 1,\ x_0 = 4$. See Fig.12.

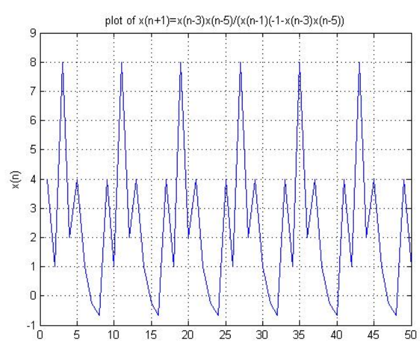

FIGURE 12. Eq. (5.1) has period eight solutions.

6. Conclusion

We found solutions for four difference equations in theorem 2.1, theorem 3.1, and theorem 4.1. and theorem 5.1, respectively. On the four difference equations, the dynamics of its behavior were studied. theorem 2.2, theorem 3.2, theorem 4.2 and theorem 5.2 stated the condition of the fixed point to be not locally asymptotic stable. Hence, we analyzed the behavior of the solutions of the difference equations Eq. (3.1) in Lemma 3.1, lemma 3.2 has periodic solutions of periods four and eight, respectively. Also, Eq. (5.1) in lemma 5.1 and lemma 5.2 has periodic solutions of period four and eight, respectively. For verification, numerical simulation was used and figures 1,2,3,4,5,6,7,8,9,10,11,12 confirmed our results. In future, we will to study these equations with periodic coefficients.

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