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PROPERTIES OF STRONGLY PRE-OPEN SETS IN IDEAL NANO TOPOLOGICAL SPACES

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ABSTRACT. Aim of this article, Rajasekaran [11] introduced strongly pre-I-open sets and in nano topological spaces. The relationships of strongly pre-nI-open sets with various other nano $\mathcal{R}_I$-set and nano I-locally closed sets are investigated.

1. INTRODUCTION

An ideal I [13] on a space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions.

(1) $A \in I$ and $B \subset A$ imply $B \in I$ and
(2) $A \in I$ and $B \in I$ imply $A \cup B \in I$.

Given a space (X, τ) with an ideal I on X if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^\star : \wp(X) \to \wp(X)$, called a local function of A with respect to τ and I is defined as follows: for $A \subset X$, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ [1]. The closure operator defined by $cl^\star(A) = A \cup A^\star(I, \tau)$ [12] is a Kuratowski closure operator which generates a topology $\tau^\star(I, \tau)$ called the $\star$-topology which is finer then τ. We will simply write $A^\star$ for $A^\star(I, \tau)$ and $\tau^\star$ for $\tau^\star(I, \tau)$. If I is an ideal on X, then (X, τ, I) is called an ideal topological space or an ideal space.

Rajasekaran et.al [8] introduced pre-nI-open sets and α-nI-open sets in the concept of ideal nano topological spaces.

In this paper, Rajasekaran [11] introduced strongly pre-I-open sets and in nano topological spaces. The relationships of strongly pre-nI-open sets with various other nano $\mathcal{R}_I$-set and nano I-locally closed sets are investigated.

2. PRELIMINARIES

Definition 2.1. [7] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to

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the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

(1) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.
(2) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.
(3) The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. [2] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then R(X) satisfies the following axioms:

(1) U and $\phi \in \tau_R(X)$,
(2) The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,
(3) The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

Thus $\tau_R(X)$ is a topology on U called the nano topology with respect to X and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed.

In the rest of the paper, we denote a nano topological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset O of U are denoted by $I_n(O)$ and $C_n(O)$, respectively.

A nano topological space $(U, \mathcal{N})$ with an ideal I on U is called [4] an ideal nano topological space and is denoted by $(U, \mathcal{N}, I)$. $G_n(x) = \{G_n \mid x \in G_n, G_n \in \mathcal{N}\}$, denotes [4] the family of nano open sets containing x.

In future an ideal nano topological spaces $(U, \mathcal{N}, I)$ is referred as a space.

Definition 2.3. [4] Let $(U, \mathcal{N}, I)$ be a space with an ideal I on U. Let $(.)^\star_n$ be a set operator from $\wp(U)$ to $\wp(U)$ ($\wp(U)$ is the set of all subsets of U).

For a subset $O \subseteq U$, $O^\star_n(I, \mathcal{N}) = \{x \in U : G_n \cap O \notin I$, for every $G_n \in G_n(x)\}$ is called the nano local function (briefly, n-local function) of A with respect to I and $\mathcal{N}$. We will simply write $O^\star_n$ for $O^\star_n(I, \mathcal{N})$.

Theorem 2.1. [4] *Let $(U, \mathcal{N}, I)$ be a space and O and B be subsets of U. Then*

(1) $O \subseteq B \Rightarrow O^\star_n \subseteq B^\star_n$,
(2) $O^\star_n = C_n(O^\star_n) \subseteq C_n(O)$ ($O^\star_n$ *is a n-closed subset of* $C_n(O)$),
(3) $(O^\star_n)^\star_n \subseteq O^\star_n$,
(4) $(O \cup B)^\star_n = O^\star_n \cup B^\star_n$,
(5) $V \in \mathcal{N} \Rightarrow V \cap O^\star_n = V \cap (V \cap O)^\star_n \subseteq (V \cap O)^\star_n$,
(6) $J \in I \Rightarrow (O \cup J)^\star_n = O^\star_n = (O - J)^\star_n$.

Theorem 2.2. [4] *Let $(U, \mathcal{N}, I)$ be a space with an ideal I and $O \subseteq O^\star_n$, then* $O^\star_n = C_n(O^\star_n) = C_n(O)$.

Definition 2.4. [6] A subset A of a space $(U, \mathcal{N}, I)$ is $n\star$-dense in itself (resp. $n\star$-perfect and $n\star$-closed) if $O \subseteq O^\star_n$ (resp. $O = O^\star_n$, $O^\star_n \subseteq O$).

The complement of a $n\star$-closed set is said to be $n\star$-open.

Definition 2.5. [3] A subset O of U in a nano topological space $(U, \mathcal{N})$ is called nano-codense (briefly n-codense) if $U - O$ is n-dense.

Theorem 2.3. [4] *Let $(U, \mathcal{N}, I)$ be an ideal nano space. Then is $\mathcal{I}$ is n-codense $if and only if O \subseteq O^\star$ for every n-open set O.*

Definition 2.6. [4] Let $(U, \mathcal{N}, I)$ be a space. The set operator $C_n^\star$ called a nano $\star$-closure is defined by $C_n^\star(O) = O \cup O_n^\star$ for $O \subseteq U$.

It can be easily observed that $C_n^\star(O) \subseteq C_n(O)$.

Theorem 2.4. [5] *In a space $(U, \mathcal{N}, I)$, if O and B are subsets of U, then the following results are true for the set operator n-$cl^\star$.*

(1) $O \subseteq C_n^\star(O)$,
(2) $C_n^\star(\phi) = \phi$ *and* $C_n^\star(U) = U$,
(3) $If O \subset B$, *then* $C_n^\star(O) \subseteq C_n^\star(B)$,
(4) $C_n^\star(O) \cup C_n^\star(B) = C_n^\star(O \cup B)$.
(5) $C_n^\star(C_n^\star(O)) = C_n^\star(O)$.

Definition 2.7. A subset O of a space $(U, \mathcal{N})$, is called a

(1) nano α-I-open (resp. α-nI-open) [8] if $O \subseteq I_n(C_n^\star(I_n(O)))$.
(2) nano pre-I-open (resp. pre-nI-open) [8] if $O \subseteq I_n(C_n^\star(O))$.
(3) nano t_α-I-set (resp. t_α-nI-set) [10] if $I_n(O) = I_n(C_n^\star(O))$.
(4) nano $\mathcal{R}_\alpha$-I-set (resp. $\mathcal{R}_\alpha$-nI-set) [10] if $O = S \cap K$, where S is n-open and K is t_α-nI-set.
(5) nano δ-I-open (resp. δ-nI-open) [9] if $I_n(C_n^\star(O)) \subseteq C_n^\star(I_n(O))$.
(6) nano $\mathcal{R}_I$-set (or) nano $\mathcal{O}_I$-set (resp. $n\mathcal{R}_I$-set) [11] if $O = S \cap K$ where S is n-open and K is $(I_n(K))_n^\star$.
(7) nano I-locally closed (resp. nI-locally closed) [11] if $O = S \cap K$ where S is n-open and K is $n\star$-perfect.

Note : The largest pre-nI-open (α-nI-open) set contained in O, denoted by $p\Im\mathrm{I}_n(O)$ ($\alpha\Im I_n(O)$), is called the pre-nI-interior (α-nI-interior).

3. Properties of strongly pre-open sets in ideal nano spaces

Definition 3.1. A subset O of an ideal nano space $(U, \mathcal{N}, I)$, is called a strongly nano pre-I-open (resp. $\mathcal{SP}$-nI-set) [11] if O is pre-nI-open and $\mathcal{R}_\alpha$-nI-set

Example 3.1. *Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{d\}, \{b, c\}\}$ and $X = \{a, c\}$. Then $\mathcal{N} = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, U\}$. Let the ideal be $I = \{\phi, \{c\}\}$.*

Theorem 3.2. *If O is any subset of an ideal nano space, then the next conditions are holds.*

(1) $p\Im I_n(O) = O \cap I_n(C_n^\star(O))$.
(2) $\alpha\Im I_n(O) = O \cap I_n(C_n^\star(I_n(O)))$.

Proof.

(1) Since
$$\begin{aligned} O \cap I_n(C_n^\star(O)) &\subseteq I_n(C^\star(O)) \\ &= I_n(I_n(C_n^\star(O))) \\ &= I_n(C_n^\star(O) \cap I_n(C_n^\star(O))) \\ &\subseteq I_n(C_n^\star(O \cap I_n(C_n^\star(O)))), \end{aligned}$$
$O \cap I_n(C^\star(O))$ is a pre-nI-open set contained in O and so
$O \cap I_n(C_n^\star(O)) \subseteq p\Im I_n(O)$.

Since $p\Im I_n(O)$ is pre-nI-open, $p\Im I_n(O) \subseteq I_n(C_n^\star(p\Im I_n(O))) \subseteq I_n(C_n^\star(O))$ and so $p\Im I_n(O) \subseteq O \cap I_n(C_n^\star(O))$.

Hence $p\Im I_n(O) = O \cap I_n(C_n^\star(O))$.

(2) Since $O \cap I_n(C_n^\star(I_n(O))) \subseteq I_n(C_n^\star(I_n(O))) = I_n(I_n(C_n^\star(I_n(O))))$

$$\begin{aligned} &= I_n(C_n^\star(I_n(O) \cap I_n(C_n^\star(I_n(O))))) \\ &\subseteq I_n(C_n^\star(I_n(O) \cap I_n(C_n^\star(I_n(O))))) \\ &= I_n(C_n^\star(I_n(O \cap I_n(C_n^\star(I_n(O)))))), \end{aligned}$$

$O \cap I_n(C_n^\star(I_n(O)))$ is an α-nI-open set contained in O and so

$$O \cap I_n(C_n^\star(I_n(O))) \subseteq \alpha\Im I_n(O).$$

Since $\alpha\Im I_n(O)$ is α-nI-open,

$$\alpha\Im I_n(O) \subseteq I_n(C_n^\star(I_n(\alpha\Im I_n(O)))) \subseteq I_n(C_n^\star(I_n(O)))$$

and so $\alpha\Im I_n(O) \subseteq O \cap I_n(C_n^\star(I_n(O)))$.

Hence $\alpha\Im I_n(O) = O \cap I_n(C_n^\star(I_n(O)))$.

□

Theorem 3.3. *If O is a $\mathcal{R}_\alpha$-nI-set of an ideal nano space, then $\alpha\Im I_n(O) = I_n(O)$.*

Proof.

Forever, $\alpha\Im I_n(O) \supseteq I_n(O)$. Since O is a $\mathcal{R}_\alpha$-nI-set, $O = S \cap K$ where S is n-open and $I_n(C_n^\star(I_n(K))) = I_n(K)$. Now

$O \subseteq K$ implies $I_n(C_n^\star(I_n(O))) \subseteq I_n(C_n^\star(I_n(K))) = I_n(K)$.

Therefore, by Theorem 3.2(1), $\alpha\Im I_n(O) = O \cap I_n(C_n^\star(I_n(O))) \subseteq O \cap I_n(K) = S \cap I_n(K) = I_n(S \cap K) = I_n(O)$. Therefore, $\alpha\Im I_n(O) \subseteq I_n(O)$, and so $\alpha\Im I_n(O) = I_n(O)$. □

Theorem 3.4. *If O is a δ-nI-open set of an ideal nano space, then $\alpha\Im I_n(O) = p\Im I_n(O)$.*

Proof.

Since every α-nI-open set is a pre-nI-open set,

$\alpha\Im I_n(O) \subseteq p\Im I_n(O)$. By Theorem 3.2(1), $\alpha\Im I_n(O) = O \cap I_n(C_n^\star(I_n(O)))$.

Since O is δ-nI-open, $\alpha\Im I_n(O) \supseteq O \cap I_n(I_n(C_n^\star(O))) = O \cap I_n(C_n^\star(O)) = p\Im I_n(O)$ and so $\alpha\Im I_n(O) \supseteq p\Im I_n(O)$. Therefore, $\alpha\Im I_n(O) = p\Im I_n(O)$.

□

Theorem 3.5. *If $(U, \mathcal{N}, I)$ is any ideal nano space, then the following conditions are holds.*

(1) If O is $n\mathcal{R}_I$-set, then O is a semi-nI-open.
(2) If O is semi-nI-open set, then O is a δ-nI-open.

Proof.

(1) If O is $n\mathcal{R}_I$-set, then $O = S \cap K$ where S is n-open and $K = (I_n(K))_n^\star$. Therefore, $O = S \cap K = S \cap (I_n(K))_n^\star \subseteq (S \cap I_n(K))_n^\star = (I_n(S \cap K))_n^\star = (I_n(O))_n^\star \subseteq C_n^\star(I_n(O))$ and so O is semi-nI-open.
(2) If O is semi-nI-open, then $O \subseteq C_n^\star(I_n(O))$. Now $I_n(C_n^\star(O) \subseteq I_n(C_n^\star(C_n^\star(I_n(O)))) = I_n(C_n^\star(I_n(O))) \subseteq C_n^\star(I_n(O)) \Longrightarrow O$ is a δ-nI-open set.

□

Remark. *If an ideal nano space,*

(1) pre-nI-open set and $n\mathcal{R}_I$-set.
(2) every n-open set is $\mathcal{SP}$-nI-set.

(3) *every $\mathcal{R}_\alpha$-nI-set is pre-nI-open.*

Example 3.6. *The Example 3.1,*

(1) *the set $\{a, d\}$ is $n\mathcal{R}_I$-set but not pre-nI-open*
(2) *the set $\{c\}$ is $\mathcal{SP}$-nI-set is not n-open.*
(3) *the set $\{b\}$ is pre-nI-open but not $\mathcal{R}_\alpha$-nI-set.*

Remark. *If an ideal nano space,*

(1) *pre-nI-open sets and $\mathcal{R}_\alpha$-nI-sets are independent.*
(2) *pre-nI-open sets and δ-nI-open sets are independent.*

Example 3.7. *The Example 3.1,*

(1) *the set $\{b\}$ is pre-nI-open but not $\mathcal{R}_\alpha$-nI-set.*
(2) *the set $\{d\}$ is $\mathcal{R}_\alpha$-nI-set but not pre-nI-open.*
(3) *the set $\{c$ is δ-nI-open but not -nI-open.*
(4) *the set $\{b\}$ is pre-nI-open but not δ-nI-open.*

Theorem 3.8. *If $(U, \mathcal{N}, I)$ is any ideal nano space and $O \subseteq U$, then the following conditions are equivalent.*

(1) *O is n-open.*
(2) *O is both $\mathcal{SP}$-nI-open and δ-nI-open.*

Proof.
(1) $\Longrightarrow$ (2) is clear.
(2) $\Longrightarrow$ (1) Suppose O is $\mathcal{SP}$-nI-open and also a δ-nI-open set. By Theorem 3.4, since O is pre-nI-open, $\alpha\Im I_n(O) = p\Im I_n(O) = O$. By Theorem 3.3, $\alpha\Im I_n(O) = I_n(O)$. Therefore, $O = I_n(O) \Longrightarrow O$ is n-open.

□

Theorem 3.9. *If $(U, \mathcal{N}, I)$ is any ideal nano space where I is n-codense and $O \subseteq U$, then the following conditions are equivalent.*

(1) *O is n-open.*
(2) *O is α-nI-open and nI-locally closed set.*
(3) *O is pre-nI-open and nI-locally closed set.*
(4) *O is pre-nI-open set and $n\mathcal{R}_I$-set.*
(5) *O is $\mathcal{SP}$-nI-open and $n\mathcal{R}_I$-set.*
(6) *O is $\mathcal{SP}$-nI-open and semi-nI-open set.*
(7) *O is $\mathcal{SP}$-nI-open and δ-nI-open set.*

Proof.
(1) $\Longrightarrow$ (2) If O is n-open, then O is α-nI-open. Since I is n-codense, by Theorem 2.3, $O \subseteq O_n^\star$, and so $O = O \cap O_n^\star$. Therefore, O is nI-locally closed.
(2) $\Longrightarrow$ (3) Follows from the fact that every α-nI-open set is pre-nI-open.
(3) $\Longrightarrow$ (4) If O is nI-locally closed, then $O = S \cap O_n^\star$ for some n-open set S. Since $O \subseteq O_n^\star$, by Theorem 2.2, $O_n^\star = C_n^\star(O)$. Since O is pre-nI-open, $O \subseteq I_n(C_n^\star(O) = I_n(O_n^\star)$ and so $O_n^\star \subseteq I_n(O_n^\star)_n^\star \subseteq (O_n^\star)_n^\star \subseteq O_n^\star$. Therefore, $O_n^\star = I_n(O_n^\star))_n^\star$ which implies that O is an $n\mathcal{R}_I$-set.
(4) $\Longrightarrow$ (5) If A is an $n\mathcal{R}_I$-set, then $O = S \cap K$ where S is n-open and $K = (I_n(K))_n^\star$. Now $I_n(C_n^\star(I_n(K))) = I_n(I_n(K) \cup (I_n(K))_n^\star) = I_n(I_n(K) \cup K) = I_n(K)$. It follows that O is $\mathcal{SP}$-nI-open.
(5) $\Longrightarrow$ (6) Follows from Theorem 3.5(1).

$(6) \implies (7)$ Follows from Theorem 3.5(2).
$(7) \implies (1)$ Follows from Theorem 3.8. □

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