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SEQUENTIAL GENERALIZED FRACTIONAL OSTROWSKI AND GRÜSS TYPE INEQUALITIES FOR SEVERAL BANACH ALGEBRA VALUED FUNCTIONS

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ABSTRACT. Employing sequential generalized Caputo fractional left and right vectorial Taylor formulae we establish mixed sequential generalized fractional Ostrowski and Grüss type inequalities for several Banach algebra valued functions. The estimates are with respect to all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$. We finish with applications.

1. INTRODUCTION

The following results motivate our work.

Theorem 1.1. *(1938, Ostrowski [10]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \tag{1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

Theorem 1.2. *(1882, Čebyšev [6]) Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a,b])$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x)\,dx - \left(\frac{1}{b-a} \int_a^b f(x)\,dx \right) \left(\frac{1}{b-a} \int_a^b g(x)\,dx \right) \right|$$
$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty. \tag{2}$$

The above integrals are assumed to exist.

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The related Grüss type inequalities have many applications to Probability Theory.

In this article we present sequential generalized fractional Ostrowski and Cebysev-Grüss type inequalities for several Banach algebra valued functions. Now our sequential generalized left and right Caputo fractional derivatives are for Banach space valued functions and our integrals are of Bochner type. The main motivation here is Theorem 2.5 at the end of section 2. Several applications finish this article. Inspiration came also from [7], [8]. See also [2]-[4].

2. Vectorial sequential generalized fractional calculus background

We need

Definition 2.1. ([5], p. 106) Let $0 < \alpha \leq 1$, $f \in C^1([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$.

We define the left generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{a+;g}^{\alpha} f\right)(x) := \frac{1}{\Gamma(1-\alpha)} \int_a^x (g(x) - g(t))^{-\alpha} g'(t) \left(f \circ g^{-1}\right)'(g(t))\, dt, \quad (3)$$

$\forall\, x \in [a,b]$, where Γ is the gamma function. The last integral is of Bochner type ([9]).

If $0 < \alpha < 1$, by Theorem 4.10, p. 98, [5], we have that $\left(D_{a+;g}^{\alpha} f\right) \in C([a,b], X)$.

We set

$$D_{a+;g}^{1} f(x) := \left(\left(f \circ g^{-1}\right)' \circ g\right)(x) \in C([a,b], X), \quad (4)$$

$$D_{a+;g}^{0} f(x) = f(x), \ \ \forall\, x \in [a,b].$$

When $g = id$, then

$$D_{a+;g}^{\alpha} f = D_{a+;id}^{\alpha} f = D_{*a}^{\alpha} f, \quad (5)$$

the usual left X-valued Caputo fractional derivative, see [5], Ch. 1.

We make

Remark. *By (3) we have*

$$\left\|\left(D_{a+;g}^{\alpha} f\right)(x)\right\| \leq \frac{1}{\Gamma(1-\alpha)} \int_a^x (g(x) - g(t))^{-\alpha} g'(t) \left\|\left(f \circ g^{-1}\right)'(g(t))\right\| dt \leq$$

$$\frac{\left\|\left\|\left(f \circ g^{-1}\right)' \circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)} \int_a^x (g(x) - g(t))^{-\alpha} g'(t)\, dt =$$

$$\frac{\left\|\left\|\left(f \circ g^{-1}\right)' \circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)} \frac{(g(x) - g(t))^{1-\alpha}}{1-\alpha} = \quad (6)$$

$$\frac{\left\|\left\|\left(f \circ g^{-1}\right)' \circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)} (g(x) - g(a))^{1-\alpha}, \ \ \forall\, x \in [a,b].$$

Hence

$$\left(D_{a+;g}^{\alpha} f\right)(a) = 0. \quad (7)$$

We need

Definition 2.2. ([5], p. 107) Let $0 < \alpha \leq 1$, $f \in C^1([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$.

We define the right generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{b-;g}^{\alpha}f\right)(x) := \frac{-1}{\Gamma(1-\alpha)}\int_x^b (g(t)-g(x))^{-\alpha} g'(t)\left(f\circ g^{-1}\right)'(g(t))\,dt, \tag{8}$$

$\forall\, x\in[a,b]$. The last integral is of Bochner type.

If $0<\alpha<1$, by Theorem 4.11, p. 101 ([5]), we have that $\left(D_{b-;g}^{\alpha}f\right)\in C([a,b],X)$.

We set

$$D_{b-;g}^{1}f(x) := -\left(\left(f\circ g^{-1}\right)'\circ g\right)(x)\in C([a,b],X), \tag{9}$$
$$D_{b-;g}^{0}f(x) := f(x),\ \ \forall\, x\in[a,b].$$

When $g=id$, then

$$D_{b-;g}^{\alpha}f(x) = D_{b-;id}^{\alpha}f(x) = D_{b-}^{\alpha}f, \tag{10}$$

the usual right X-valued Caputo fractional derivative, see [5], Ch. 2.

We make

Remark. *By (8) we have*

$$\left\|\left(D_{b-;g}^{\alpha}f\right)(x)\right\| \leq \frac{1}{\Gamma(1-\alpha)}\int_x^b (g(t)-g(x))^{-\alpha} g'(t)\left\|\left(f\circ g^{-1}\right)'(g(t))\right\| dt \leq$$
$$\frac{\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)}\int_x^b (g(t)-g(x))^{-\alpha} g'(t)\,dt = \tag{11}$$
$$\frac{\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(1-\alpha)}\frac{(g(b)-g(x))^{1-\alpha}}{1-\alpha} =$$
$$\frac{\left\|\left\|\left(f\circ g^{-1}\right)'\circ g\right\|\right\|_{\infty,[a,b]}}{\Gamma(2-\alpha)}(g(b)-g(x))^{1-\alpha},\ \ \forall\, x\in[a,b].$$

Hence

$$\left(D_{b-;g}^{\alpha}f\right)(b) = 0. \tag{12}$$

We need

Definition 2.3. ([5], p. 115) Denote by $(0<\alpha\leq 1)$

$$D_{a+;g}^{n\alpha} := D_{a+;g}^{\alpha}D_{a+;g}^{\alpha}...D_{a+;g}^{\alpha}\ \ (n\text{ times}),\ n\in\mathbb{N} \tag{13}$$

and $D_{a+;g}^{0}=I$ (identity operator).

We also need

Definition 2.4. ([5], p. 118)

$$D_{b-;g}^{n\alpha} := D_{b-;g}^{\alpha}D_{b-;g}^{\alpha}...D_{b-;g}^{\alpha}\ \ (n\text{ times}),\ n\in\mathbb{N} \tag{14}$$

and $D_{b-;g}^{0}=I$ (identity operator).

Based on (7) and Theorem 4.30, p. 117, ([5]), we have the following g-left generalized modified X-valued Taylor's formula:

Theorem 2.1. *Let $0<\alpha\leq 1$, $n\in\mathbb{N}$, $f\in C^1([a,b],X)$, $(X,\|\cdot\|)$ a Banach space, $g\in C^1([a,b])$, strictly increasing, such that $g^{-1}\in C^1([g(a),g(b)])$. Let $F_k := D_{a+;g}^{k\alpha}f$,*

$k = 1, ..., n$, *that fulfill* $F_k \in C^1([a,b], X)$, *and* $F_{n+1} \in C([a,b], X)$. *Then*

$$f(x) - f(a) = \sum_{i=2}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{a+;g}^{i\alpha} f\right)(a) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g(x) - g(t))^{(n+1)\alpha-1} g'(t) \left(D_{a+;g}^{(n+1)\alpha} f\right)(t)\, dt, \tag{15}$$

$\forall\, x \in [a,b]$.

When $n = 1$ we obtain

Corollary 2.2. *Let* $0 < \alpha \leq 1$, $f \in C^1([a,b], X)$, $(X, \|\cdot\|)$ *is a Banach space,* $g \in C^1([a,b])$, *strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$. *Assume that* $D_{a+;g}^{\alpha} f \in C^1([a,b], X)$, *and* $D_{a+;g}^{2\alpha} f \in C([a,b], X)$. *Then*

$$f(x) - f(a) = \frac{1}{\Gamma(2\alpha)} \int_a^x (g(x) - g(t))^{2\alpha-1} g'(t) \left(D_{a+;g}^{2\alpha} f\right)(t)\, dt, \tag{16}$$

$\forall\, x \in [a,b]$.

Based on (12) and Theorem 4.33, p. 120, ([5]), we have the following g-right generalized modified X-valued Taylor's formula:

Theorem 2.3. *Let* $f \in C^1([a,b], X)$, $(X, \|\cdot\|)$ *a Banach space,* $g \in C^1([a,b])$, *strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$. *Suppose* $F_k := D_{b-;g}^{k\alpha} f$, $k = 1, ..., n$, *fulfill* $F_k \in C^1([a,b], X)$, *and* $F_{n+1} \in C([a,b], X)$, *where* $0 < \alpha \leq 1$, $n \in \mathbb{N}$. *Then*

$$f(x) - f(b) = \sum_{i=2}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-;g}^{i\alpha} f\right)(b) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (g(t) - g(x))^{(n+1)\alpha-1} g'(t) \left(D_{b-;g}^{(n+1)\alpha} f\right)(t)\, dt, \tag{17}$$

$\forall\, x \in [a,b]$.

When $n = 1$ we obtain

Corollary 2.4. *Let* $0 < \alpha \leq 1$, $f \in C^1([a,b], X)$, $(X, \|\cdot\|)$ *is a Banach space,* $g \in C^1([a,b])$, *strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$. *Assume that* $D_{b-;g}^{\alpha} f \in C^1([a,b], X)$, *and* $D_{b-;g}^{2\alpha} f \in C([a,b], X)$. *Then*

$$f(x) - f(b) = \frac{1}{\Gamma(2\alpha)} \int_x^b (g(t) - g(x))^{2\alpha-1} g'(t) \left(D_{b-;g}^{2\alpha} f\right)(t)\, dt, \tag{18}$$

$\forall\, x \in [a,b]$.

We are greatly motivated by the following sequential generalized fractional Ostrowski type inequality:

Theorem 2.5. *(p. 140,* [5]*) Let* $g \in C^1([a,b])$ *and strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$, *and* $0 < \alpha < 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, *where* $(X, \|\cdot\|)$ *is a Banach space. Let* $x_0 \in [a,b]$ *be fixed. Assume that* $F_k^{x_0} := D_{x_0-;g}^{k\alpha} f$, *for* $k = 1, ..., n$, *fulfill* $F_k^{x_0} \in C^1([a,b], X)$ *and* $F_{n+1}^{x_0} \in C([a, x_0], X)$ *and* $\left(D_{x_0-;g}^{i\alpha} f\right)(x_0) = 0$, $i = 1, ..., n$.

Similarly, we assume that $G_k^{x_0} := D_{x_0+;g}^{k\alpha} f$, *for* $k = 1, ..., n$, *fulfill* $G_k^{x_0} \in C^1([x_0, b], X)$ *and* $G_{n+1}^{x_0} \in ([x_0, b], X)$ *and* $\left(D_{x_0+;g}^{i\alpha} f\right)(x_0) = 0$, $i = 1, ..., n$.

Then

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(x_0) \right\| \leq \frac{1}{(b-a)\,\Gamma((n+1)\alpha+1)} \cdot$$
$$\left\{ (g(b)-g(x_0))^{(n+1)\alpha} (b-x_0) \left\| D_{x_0+;g}^{(n+1)\alpha} f \right\|_{\infty,[x_0,b]} + \right. \tag{19}$$
$$\left. (g(x_0)-g(a))^{(n+1)\alpha} (x_0-a) \left\| D_{x_0-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,x_0]} \right\}.$$

3. Banach Algebras background

All here come from [11].

We need

Definition 3.1. ([11], p. 245) A complex algebra is a vector space A over the complex filed $\mathbb{C}$ in which a multiplication is defined that satisfies

$$x(yz) = (xy)z, \tag{20}$$

$$(x+y)z = xz + yz,\ x(y+z) = xy + xz, \tag{21}$$

and

$$\alpha(xy) = (\alpha x) y = x(\alpha y), \tag{22}$$

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A, y \in A) \tag{23}$$

and if A contains a unit element e such that

$$xe = ex = x \ \ (x \in A) \tag{24}$$

and

$$\|e\| = 1, \tag{25}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark. *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (24).

Inequality (23) makes multiplication to be continuous, more precisely left and right continuous, see [11]*, p. 246.*

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [11]*, p. 247-248, § 10.3.*

We also make

Remark. *Next we mention about integration of A-valued functions, see* [11]*, p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [11]*, simply because A is a*

Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \, d\mu = \int_Q x f(p) \, d\mu(p) \tag{26}$$

and

$$\left(\int_Q f \, d\mu \right) x = \int_Q f(p) \, x \, d\mu(p) . \tag{27}$$

The Bochner integrals we will involve in our article follow (26) and (27). Also, let $f \in C([a,b], X)$, where $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [5], *p. 3, f is Bochner integrable.*

4. Main Results

We start with mixed sequential generalized fractional Ostrowski type inequalities for several functions over a Banach algebra. A uniform estimate follows.

Theorem 4.1. *Let $(A, \|\cdot\|)$ be a Banach algebra, $x_0 \in [a,b] \subset \mathbb{R}$, $0 < \alpha < 1$, $n \in \mathbb{N}$, $f_i \in C^1([a,b], A)$, $i = 1, ..., r$. Let $g \in C^1([a,b])$ and strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$. Assume that $F_{ki}^{x_0} := D_{x_0-;g}^{k\alpha} f_i$, for $k = 1, ..., n$, fulfill $F_{ki}^{x_0} \in C^1([a, x_0], A)$ and $F_{(n+1)i}^{x_0} \in C([a, x_0], A)$ and $\left(D_{x_0-;g}^{j\alpha} f_i \right)(x_0) = 0$, $j = 2, ..., n$; $i = 1, ..., r$. Similarly, we assume that $G_{ki}^{x_0} := D_{x_0+;g}^{k\alpha} f_i$, $k = 1, ..., n$, fulfill $G_{ki}^{x_0} \in C^1([x_0, b], A)$ and $G_{(n+1)i}^{x_0} \in C([x_0, b], A)$ and $\left(D_{x_0+;g}^{j\alpha} f_i \right)(x_0) = 0$, $j = 2, ..., n$; $i = 1, ..., r$.*

Denote by

$$\Omega(f_1, ..., f_r)(x_0) :=$$
$$\sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) \, dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0) \right] . \tag{28}$$

Then

$$\|\Omega(f_1, ..., f_r)(x_0)\| \leq \frac{1}{\Gamma((n+1)\alpha + 1)} \sum_{i=1}^{r} \left[\left[\left\| \left\| \left(D_{x_0-;g}^{(n+1)\alpha} f_i \right) \right\| \right\|_{\infty,[a,x_0]} \right. \right.$$
$$\left. (g(x_0) - g(a))^{(n+1)\alpha} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right] + \tag{29}$$
$$\left. \left[\left\| \left\| \left(D_{x_0+;g}^{(n+1)\alpha} f_i \right) \right\| \right\|_{\infty,[x_0,b]} (g(b) - g(x_0))^{(n+1)\alpha} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right] \right] .$$

Proof. By Theorem 2.3 we obtain

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma((n+1)\alpha)} \int_x^{x_0} (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left(D_{x_0-;g}^{(n+1)\alpha} f_i \right)(t) \, dt, \tag{30}$$

$\forall \, x \in [a, x_0], i = 1, ..., r.$

Also, by Theorem 2.1, we get

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma((n+1)\alpha)} \int_{x_0}^{x} (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D_{x_0+;g}^{(n+1)\alpha} f_i \right)(t)\, dt, \tag{31}$$

$\forall\, x \in [x_0, b]$, $i = 1, ..., r$.

Left multiplying (30) and (31) with $\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right)$ we get, respectively,

$$\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) - \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x_0) = \tag{32}$$

$$\frac{\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right)}{\Gamma((n+1)\alpha)} \int_{x}^{x_0} (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left(D_{x_0-;g}^{(n+1)\alpha} f_i \right)(t)\, dt,$$

$\forall\, x \in [a, x_0]$, $i = 1, ..., r$,

and

$$\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) - \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x_0) =$$

$$\frac{\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right)}{\Gamma((n+1)\alpha)} \int_{x_0}^{x} (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D_{x_0+;g}^{(n+1)\alpha} f_i \right)(t)\, dt, \tag{33}$$

$\forall\, x \in [x_0, b]$, $i = 1, ..., r$.

Adding (32) and (33) as separate groups, we obtain

$$\sum_{i=1}^{r} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x_0) =$$

$$\frac{\sum_{i=1}^{r} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right)}{\Gamma((n+1)\alpha)} \int_{x}^{x_0} (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left(D_{x_0-;g}^{(n+1)\alpha} f_i \right)(t)\, dt, \tag{34}$$

$\forall\, x \in [a, x_0]$,

and

$$\sum_{i=1}^{r} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x_0) =$$

$$\frac{\sum_{i=1}^{r} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right)}{\Gamma((n+1)\alpha)} \int_{x_0}^{x} (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D_{x_0+;g}^{(n+1)\alpha} f_i \right)(t)\, dt, \tag{35}$$

$\forall\, x \in [x_0, b]$.

Next, we integrate (34) and (35) with respect to $x \in [a,b]$. We have

$$\sum_{i=1}^{r}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \sum_{i=1}^{r}\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0) =$$

$$\frac{1}{\Gamma((n+1)\alpha)}\sum_{i=1}^{r}\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\right.$$

$$\left.\left(\int_{x}^{x_0}(g(t)-g(x))^{(n+1)\alpha-1} g'(t)\left(D_{x_0-;g}^{(n+1)\alpha} f_i\right)(t)\,dt\right)dx\right], \tag{36}$$

and

$$\sum_{i=1}^{r}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \sum_{i=1}^{r}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0) =$$

$$\frac{1}{\Gamma((n+1)\alpha)}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\right.$$

$$\left.\left(\int_{x_0}^{x}(g(x)-g(t))^{(n+1)\alpha-1} g'(t)\left(D_{x_0+;g}^{(n+1)\alpha} f_i\right)(t)\,dt\right)dx\right]. \tag{37}$$

Finally, adding (36) and (37) we obtain the useful identity

$$\Omega(f_1,\dots,f_r)(x_0) :=$$

$$\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0)\right] =$$

$$\frac{1}{\Gamma((n+1)\alpha)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\right.\right.$$

$$\left.\left(\int_{x}^{x_0}(g(t)-g(x))^{(n+1)\alpha-1} g'(t)\left(D_{x_0-;g}^{(n+1)\alpha} f_i\right)(t)\,dt\right)dx\right]$$

$$\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{x_0}^{x}(g(x)-g(t))^{(n+1)\alpha-1} g'(t)\left(D_{x_0+;g}^{(n+1)\alpha} f_i\right)(t)\,dt\right)dx\right]\right]. \tag{38}$$

Therefore, we get that

$$\|\Omega(f_1,\dots,f_r)(x_0)\| \leq \frac{1}{\Gamma((n+1)\alpha)}\sum_{i=1}^{r}\left[\left\|\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\right.\right.\right.$$

$$\left(\int_{x}^{x_0} (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left(D_{x_0-;g}^{(n+1)\alpha} f_i\right)(t)\, dt\right) dx\Bigg]\Bigg\| +$$

$$\left\|\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{x_0}^{x} (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D_{x_0+;g}^{(n+1)\alpha} f_i\right)(t)\, dt\right) dx\right]\right\|\Bigg] \tag{39}$$

$$\leq \frac{1}{\Gamma((n+1)\alpha)} \sum_{i=1}^{r} \Bigg[\Bigg[\int_{a}^{x_0} \Bigg\| \left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)$$

$$\left(\int_{x}^{x_0} (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left(D_{x_0-;g}^{(n+1)\alpha} f_i\right)(t)\, dt\right)\Bigg\| dx\Bigg] +$$

$$\left[\int_{x_0}^{b} \left\| \left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{x_0}^{x} (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D_{x_0+;g}^{(n+1)\alpha} f_i\right)(t)\, dt\right)\right\| dx\right]\Bigg]$$

$$\leq \frac{1}{\Gamma((n+1)\alpha)} \sum_{i=1}^{r} \Bigg[\Bigg[\int_{a}^{x_0} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|\right)$$

$$\left(\int_{x}^{x_0} (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left\|\left(D_{x_0-;g}^{(n+1)\alpha} f_i\right)(t)\right\| dt\right) dx\Bigg] +$$

$$\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|\right)\left(\int_{x_0}^{x} (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left\|\left(D_{x_0+;g}^{(n+1)\alpha} f_i\right)(t)\right\| dt\right) dx\right]\Bigg]$$

$$=: (*) . \tag{40}$$

Hence it holds

$$\|\Omega(f_1, ..., f_r)(x_0)\| \leq (*) . \tag{41}$$

We have that

$$(*) \leq \frac{1}{\Gamma((n+1)\alpha + 1)}$$

$$\sum_{i=1}^{r} \left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha} f_i\right)\right\|\right\|_{\infty,[a,x_0]} \int_{a}^{x_0} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|\right) (g(x_0) - g(x))^{(n+1)\alpha} dx\right]\right.$$

$$\left. + \left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha} f_i\right)\right\|\right\|_{\infty,[x_0,b]} \int_{x_0}^{b} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|\right) (g(x) - g(x_0))^{(n+1)\alpha} dx\right]\right] \leq$$

$$\frac{1}{\Gamma((n+1)\alpha + 1)} \sum_{i=1}^{r} \left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha} f_i\right)\right\|\right\|_{\infty,[a,x_0]}\right.\right.$$

$$\left.(g(x_0)-g(a))^{(n+1)\alpha}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^r\|f_j(x)\|\right)dx\right)\right]+ \tag{42}$$

$$\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}f_i\right)\right\|\right\|_{\infty,[x_0,b]}(g(b)-g(x_0))^{(n+1)\alpha}\left(\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^r\|f_j(x)\|\right)dx\right)\right]\Bigg],$$

proving (29). □

Next comes an L_1 estimate.

Theorem 4.2. *All as in Theorem 4.1, plus* $\frac{1}{(n+1)}\leq\alpha<1$, $n\in\mathbb{N}$. *Then*

$$\|\Omega(f_1,...,f_r)(x_0)\|\leq\frac{\|g'\|_{\infty,[a,b]}}{\Gamma((n+1)\alpha)}$$

$$\sum_{i=1}^r\left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}f_i\right)\right\|\right\|_{L_1([a,x_0])}(g(x_0)-g(a))^{(n+1)\alpha-1}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^r\|f_j(x)\|\right)dx\right)\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}f_i\right)\right\|\right\|_{L_1([x_0,b])}(g(b)-g(x_0))^{(n+1)\alpha-1}\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^r\|f_j(x)\|\right)dx\right]\right]. \tag{43}$$

Proof. We have that (by (40), (41))

$$(*)\leq\frac{\|g'\|_{\infty,[a,b]}}{\Gamma((n+1)\alpha)}$$

$$\sum_{i=1}^r\left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}f_i\right)\right\|\right\|_{L_1([a,x_0])}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^r\|f_j(x)\|\right)(g(x_0)-g(x))^{(n+1)\alpha-1}dx\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}f_i\right)\right\|\right\|_{L_1([x_0,b])}\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^r\|f_j(x)\|\right)(g(x)-g(x_0))^{(n+1)\alpha-1}dx\right]\right]$$

$$\leq\frac{\|g'\|_{\infty,[a,b]}}{\Gamma((n+1)\alpha)}$$

$$\sum_{i=1}^r\left[\left[\left\|\left|\left|\left(D_{x_0-;g}^{(n+1)\alpha}f_i\right)\right|\right|\right\|_{L_1([a,x_0])}(g(x_0)-g(a))^{(n+1)\alpha-1}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^r\|f_j(x)\|\right)dx\right)\right]\right. \tag{44}$$

$$\left.+\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}f_i\right)\right\|\right\|_{L_1([x_0,b])}(g(b)-g(x_0))^{(n+1)\alpha-1}\left(\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^r\|f_j(x)\|\right)dx\right)\right]\right],$$

proving the claim. □

An L_p estimate follows.

Theorem 4.3. *All as in Theorem 4.1, plus* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *and* $\frac{1}{q(n+1)}<\alpha<1$. *Then*

$$\|\Omega(f_1,...,f_r)(x_0)\|\leq\frac{1}{\Gamma((n+1)\alpha)(p((n+1)\alpha-1)+1)^{\frac{1}{p}}}$$
$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{(n+1)\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{q,[g(a),g(x_0)]}\right.\right.$$
$$\left.(g(x_0)-g(a))^{(n+1)\alpha-\frac{1}{q}}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]+$$
$$\left[\left\|\left\|\left(D_{x_0+;g}^{(n+1)\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{q,[g(x_0),g(b)]}(g(b)-g(x_0))^{(n+1)\alpha-\frac{1}{q}}\right.$$
$$\left.\left.\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right].\tag{45}$$

Proof. By (40) and (41) we obtain

$$(*)=\frac{1}{\Gamma((n+1)\alpha)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\right.\right.$$
$$\left.\left(\int_x^{x_0}(g(t)-g(x))^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0-;g}^{(n+1)\alpha}f_i\right)\circ g^{-1}\right)(g(t))\right\|dg(t)\right)dx\right]+$$
$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\right.$$
$$\left.\left.\left(\int_{x_0}^{x}(g(x)-g(t))^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0+;g}^{(n+1)\alpha}f_i\right)\circ g^{-1}\right)(g(t))\right\|dg(t)\right)dx\right]\right]\tag{46}$$
$$=\frac{1}{\Gamma((n+1)\alpha)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\right.\right.$$
$$\left.\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0-;g}^{(n+1)\alpha}f_i\right)\circ g^{-1}\right)(z)\right\|dz\right)dx\right]+$$
$$\left.\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{(n+1)\alpha-1}\left\|\left(\left(D_{x_0+;g}^{(n+1)\alpha}f_i\right)\circ g^{-1}\right)(z)\right\|dz\right)dx\right]\right]$$

$$\leq \frac{1}{\Gamma((n+1)\alpha)} \sum_{i=1}^{r} \left[\left[\int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{p((n+1)\alpha - 1)} dz \right)^{\frac{1}{p}} \right. \right.$$

$$\left. \left(\int_{g(x)}^{g(x_0)} \left\| \left(\left(D_{x_0-;g}^{(n+1)\alpha} f_i \right) \circ g^{-1} \right)(z) \right\|^q dz \right)^{\frac{1}{q}} dx \right] +$$

$$\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{p((n+1)\alpha - 1)} dz \right)^{\frac{1}{p}} \right.$$

$$\left. \left. \left(\int_{g(x_0)}^{g(x)} \left\| \left(\left(D_{x_0+;g}^{(n+1)\alpha} f_i \right) \circ g^{-1} \right)(z) \right\|^q dz \right)^{\frac{1}{q}} dx \right] \right] \leq \tag{47}$$

$$\frac{1}{\Gamma((n+1)\alpha)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r} \left[\left[\left\| \left\| \left(\left(D_{x_0-;g}^{(n+1)\alpha} f_i \right) \circ g^{-1} \right) \right\| \right\|_{q,[g(a),g(x_0)]} \right. \right.$$

$$\left. \left(\int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) (g(x_0) - g(x))^{\frac{p((n+1)\alpha - 1) + 1}{p}} dx \right) \right] +$$

$$\left[\left\| \left\| \left(\left(D_{x_0+;g}^{(n+1)\alpha} f_i \right) \circ g^{-1} \right) \right\| \right\|_{q,[g(x_0),g(b)]} \right.$$

$$\left. \left. \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) (g(x) - g(x_0))^{\frac{p((n+1)\alpha - 1) + 1}{p}} dx \right) \right] \right]$$

$$\leq \frac{1}{\Gamma((n+1)\alpha)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r} \left[\left\| \left\| \left(D_{x_0-;g}^{(n+1)\alpha} f_i \right) \circ g^{-1} \right\| \right\|_{q,[g(a),g(x_0)]} \right.$$

$$(g(x_0) - g(a))^{(n+1)\alpha - \frac{1}{q}} \left(\int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right)$$

$$+ \left\| \left\| \left(D_{x_0+;g}^{(n+1)\alpha} f_i \right) \circ g^{-1} \right\| \right\|_{q,[g(x_0),g(b)]} (g(b) - g(x_0))^{(n+1)\alpha - \frac{1}{q}}$$

$$\left. \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right], \tag{48}$$

proving the claim. □

When $n = 1$ we get the following results:

Corollary 4.4. *(to Theorem 4.1) Let $(A, \|\cdot\|)$ be a Banach algebra, $x_0 \in [a,b] \subset \mathbb{R}$, $0 < \alpha < 1$, $f_i \in C^1([a,b], A)$, $i = 1, ..., r$. Let $g \in C^1([a,b])$ and strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$. Assume that $D^{\alpha}_{x_0-;g} f_i \in C^1([a,x_0], A)$ and $D^{2\alpha}_{x_0-;g} f_i \in C([a,x_0], A)$; $i = 1, ..., r$. Similarly, assume that $D^{\alpha}_{x_0+;g} f_i \in C^1([x_0,b], A)$ and $D^{2\alpha}_{x_0+;g} f_i \in C([x_0,b], A)$; $i = 1, ..., r$.*

Denote by

$$\Omega(f_1, ..., f_r)(x_0) := \sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x)\, dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0) \right]. \tag{49}$$

Then

$$\|\Omega(f_1, ..., f_r)(x_0)\| \leq \frac{1}{\Gamma(2\alpha+1)} \sum_{i=1}^{r} \left[\left[\left\| \left\| \left(D^{2\alpha}_{x_0-;g} f_i \right) \right\| \right\|_{\infty,[a,x_0]} (g(x_0) - g(a))^{2\alpha} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right] + \right. \tag{50}$$

$$\left. \left[\left\| \left\| \left(D^{2\alpha}_{x_0+;g} f_i \right) \right\| \right\|_{\infty,[x_0,b]} (g(b) - g(x_0))^{2\alpha} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right] \right] =: \Lambda_1(x_0).$$

Proof. By Theorem 4.1 and Corollaries 2.2, 2.4. □

We continue with

Corollary 4.5. *(to Theorem 4.2) All as in Corollary 4.4, plus $\frac{1}{2} \leq \alpha < 1$. Then*

$$\|\Omega(f_1, ..., f_r)(x_0)\| \leq \frac{\|g'\|_{\infty,[a,b]}}{\Gamma(2\alpha)} \sum_{i=1}^{r} \left[\left[\left\| \left\| \left(D^{2\alpha}_{x_0-;g} f_i \right) \right\| \right\|_{L_1([a,x_0])} (g(x_0) - g(a))^{2\alpha-1} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right] + \right.$$

$$\left. \left[\left\| \left\| \left(D^{2\alpha}_{x_0+;g} f_i \right) \right\| \right\|_{L_1([x_0,b])} (g(b) - g(x_0))^{2\alpha-1} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right] \right] = \Lambda_2(x_0). \tag{51}$$

Proof. By Theorem 4.2. □

We also give

Corollary 4.6. *(to Theorem 4.3) All as in Corollary 4.4, plus $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\frac{1}{2q} < \alpha < 1$. Then*

$$\|\Omega(f_1, ..., f_r)(x_0)\| \leq \frac{1}{\Gamma(2\alpha)(p(2\alpha-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{2\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{q,[g(a),g(x_0)]}\right.\right.$$
$$\left.(g(x_0)-g(a))^{2\alpha-\frac{1}{q}}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]+$$
$$\left[\left\|\left\|\left(D_{x_0+;g}^{2\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{q,[g(x_0),g(b)]}(g(b)-g(x_0))^{2\alpha-\frac{1}{q}}\right.$$
$$\left.\left.\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right]=\Lambda_3(x_0). \tag{52}$$

Proof. By Theorem 4.3. □

We make

Remark. *1) Call and assume*

$$N_1(f_1,...,f_r):=\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D_{x_0-;g}^{2\alpha}f_i\right)\right\|\right\|_{\infty,[a,x_0]},\right.$$
$$\left.\sup_{x_0\in[a,b]}\left\|\left\|\left(D_{x_0+;g}^{2\alpha}f_i\right)\right\|\right\|_{\infty,[x_0,b]}\right\}<\infty. \tag{53}$$

Then (by (50))

$$\Lambda_1(x_0)\leq\frac{(g(b)-g(a))^{2\alpha}N_1(f_1,...,f_r)}{\Gamma(2\alpha+1)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right). \tag{54}$$

2) Call and assume

$$N_2(f_1,...,f_r):=\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D_{x_0-;g}^{2\alpha}f_i\right)\right\|\right\|_{L_1([a,x_0])},\right.$$
$$\left.\sup_{x_0\in[a,b]}\left\|\left\|\left(D_{x_0+;g}^{2\alpha}f_i\right)\right\|\right\|_{L_1([x_0,b])}\right\}<\infty. \tag{55}$$

Then (by (51))

$$\Lambda_2(x_0)\leq\frac{\|g'\|_{\infty,[a,b]}(g(b)-g(a))^{2\alpha-1}N_2(f_1,...,f_r)}{\Gamma(2\alpha)}$$
$$\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right). \tag{56}$$

3). Call and assume

$$N_3(f_1,...,f_r):=\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|\left(D_{x_0-;g}^{2\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{q,[g(a),g(x_0)]},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|\left(D^{2\alpha}_{x_0+;g}f_i\right)\circ g^{-1}\right\|\right\|_{q,[g(x_0),g(b)]}\right\}<\infty. \tag{57}$$

Then (by (52))

$$\Lambda_3(x_0)\leq\frac{(g(b)-g(a))^{2\alpha-\frac{1}{q}}N_3(f_1,...,f_r)}{\Gamma(2\alpha)(p(2\alpha-1)+1)^{\frac{1}{p}}}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right). \tag{58}$$

Next denote by

$$\Phi(f_1,...,f_r):=\int_a^b\Omega(f_1,...,f_r)(x_0)\,dx_0=$$
$$\sum_{i=1}^{r}\left[(b-a)\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x)\,dx\right)-\right.$$
$$\left.\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)\left(\int_a^b f_i(x)\,dx\right)\right]. \tag{59}$$

Furthermore, it holds (see (50)-(52))

$$\|\Phi(f_1,...,f_r)\|\leq\int_a^b\|\Omega(f_1,...,f_r)(x_0)\|\,dx_0\leq\int_a^b\Lambda_i(x_0)\,dx_0,\quad i=1,2,3, \tag{60}$$

under proper assumptions, respectively, and

$$\|\Phi(f_1,...,f_r)\|=$$
$$\left\|\sum_{i=1}^{r}\left[(b-a)\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x)\,dx\right)-\right.\right.$$
$$\left.\left.\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)\left(\int_a^b f_i(x)\,dx\right)\right]\right\|. \tag{61}$$

Therefore, by (54) we get that

$$\|\Phi(f_1,...,f_r)\|\leq\frac{(b-a)(g(b)-g(a))^{2\alpha}N_1(f_1,...,f_r)}{\Gamma(2\alpha+1)}$$
$$\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right), \tag{62}$$

and by (56) we obtain

$$\|\Phi(f_1,...,f_r)\|\leq\frac{\|g'\|_{\infty,[a,b]}(b-a)(g(b)-g(a))^{2\alpha-1}N_2(f_1,...,f_r)}{\Gamma(2\alpha)}$$

$$\sum_{i=1}^{r}\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\Vert f_{j}\left(x\right)\right\Vert\right)dx\right), \tag{63}$$

and finally by (58) we derive that

$$\left\Vert \Phi\left(f_{1},...,f_{r}\right)\right\Vert \leq \frac{\left(b-a\right)\left(g\left(b\right)-g\left(a\right)\right)^{2\alpha-\frac{1}{q}}N_{3}\left(f_{1},...,f_{r}\right)}{\Gamma\left(2\alpha\right)\left(p\left(2\alpha-1\right)+1\right)^{\frac{1}{p}}}$$
$$\sum_{i=1}^{r}\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\Vert f_{j}\left(x\right)\right\Vert\right)dx\right), \tag{64}$$

all the above are valid under proper assumptions, respectively.

We have proved the following generalized Grüss-Cebysev type inequalities for several functions over a Banach algebra. A uniform estimate follows.

Theorem 4.7. *Let $(A,\|\cdot\|)$ be a Banach algebra, $[a,b]\subset\mathbb{R}$, $0<\alpha<1$, $f_i\in C^1([a,b],A)$, $i=1,...,r$. Let $g\in C^1([a,b])$ and strictly increasing, such that $g^{-1}\in C^1([g(a),g(b)])$. Assume that for every $x_0\in[a,b]$, $D^{\alpha}_{x_0-;g}f_i\in C^1([a,x_0],A)$ and $D^{2\alpha}_{x_0-;g}f_i\in C([a,x_0],A)$; $i=1,...,r$. Similarly, assume that for every $x_0\in[a,b]$, $D^{\alpha}_{x_0+;g}f_i\in C^1([x_0,b],A)$ and $D^{2\alpha}_{x_0+;g}f_i\in C([x_0,b],A)$; $i=1,...,r$. Here $N_1(f_1,...,f_r)$ is as in (53). Then*

$$\left\Vert\sum_{i=1}^{r}\left[(b-a)\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}f_{j}(x)\right)f_{i}(x)\,dx\right)-\right.\right.$$
$$\left.\left.\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}f_{j}(x)\right)dx\right)\left(\int_{a}^{b}f_{i}(x)\,dx\right)\right]\right\Vert\leq$$
$$\frac{(b-a)\left(g(b)-g(a)\right)^{2\alpha}N_{1}(f_{1},...,f_{r})}{\Gamma(2\alpha+1)}\sum_{i=1}^{r}\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\Vert f_{j}(x)\right\Vert\right)dx\right). \tag{65}$$

An L_1 estimate follows.

Theorem 4.8. *All as in Theorem 4.7, $\frac{1}{2}\leq\alpha<1$. Here $N_2(f_1,...,f_r)$ is as in (55). Then*

$$\left\Vert\sum_{i=1}^{r}\left[(b-a)\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}f_{j}(x)\right)f_{i}(x)\,dx\right)-\right.\right.$$
$$\left.\left.\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}f_{j}(x)\right)dx\right)\left(\int_{a}^{b}f_{i}(x)\,dx\right)\right]\right\Vert\leq$$

$$\frac{\|g'\|_{\infty,[a,b]}(b-a)(g(b)-g(a))^{2\alpha-1}N_2(f_1,...,f_r)}{\Gamma(2\alpha)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right). \tag{66}$$

An L_p estimate follows.

Theorem 4.9. *All as in Theorem 4.7, $p,q>1:\frac{1}{p}+\frac{1}{q}=1$ and $\frac{1}{2q}<\alpha<1$. Here $N_3(f_1,...,f_r)$ is as in (57). Then*

$$\left\|\sum_{i=1}^{r}\left[(b-a)\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x)\,dx\right)-\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)\left(\int_a^b f_i(x)\,dx\right)\right]\right\|\leq$$

$$\frac{(b-a)(g(b)-g(a))^{2\alpha-\frac{1}{q}}N_3(f_1,...,f_r)}{\Gamma(2\alpha)(p(2\alpha-1)+1)^{\frac{1}{p}}}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right). \tag{67}$$

5. Applications

We make

Remark. *From now on we assume that $(A,\|\cdot\|)$ is a commutative Banach algebra. Then, we get that*

$$\Omega(f_1,...,f_r)(x_0)\overset{(28)}{=}r\int_a^b\left(\prod_{j=1}^{r}f_j(x)\right)dx-\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)f_i(x_0), \tag{68}$$

$x_0\in[a,b]$.

Furthermore, it holds (case of $n=1$)

$$\Phi(f_1,...,f_r)\overset{(59)}{=}r(b-a)\int_a^b\left(\prod_{j=1}^{r}f_j(x)\right)dx-\sum_{i=1}^{r}\left[\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)\left(\int_a^b f_i(x)\,dx\right)\right]. \tag{69}$$

When $r=2$, we get that

$$\Omega(f_1,f_2)(x_0)=2\int_a^b f_1(x)f_2(x)\,dx-f_1(x_0)\int_a^b f_2(x)\,dx-f_2(x_0)\int_a^b f_1(x)\,dx, \tag{70}$$

$x_0\in[a,b]$,

and

$$\Phi\left(f_{1},f_{2}\right)=2\left[\left(b-a\right)\int_{a}^{b}f_{1}\left(x\right)f_{2}\left(x\right)dx-\left(\int_{a}^{b}f_{1}\left(x\right)dx\right)\left(\int_{a}^{b}f_{2}\left(x\right)dx\right)\right], \tag{71}$$

case of $n=1$.

We give

Corollary 5.1. *All as in Theorem 4.1, A is a commutative Banach algebra, $r=2$. Then*

$$\left\|\Omega\left(f_{1},f_{2}\right)\left(x_{0}\right)\right\|\leq\frac{1}{\Gamma\left(\left(n+1\right)\alpha+1\right)}\sum_{i=1}^{2}\left[\left[\left\|\left\|\left(D_{x_{0}-;g}^{(n+1)\alpha}f_{i}\right)\right\|\right\|_{\infty,[a,x_{0}]}\right.\right.$$
$$\left.\left(g\left(x_{0}\right)-g\left(a\right)\right)^{(n+1)\alpha}\left(\int_{a}^{x_{0}}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\left\|f_{j}\left(x\right)\right\|\right)dx\right)\right]+ \tag{72}$$
$$\left.\left.\left[\left\|\left\|\left(D_{x_{0}+;g}^{(n+1)\alpha}f_{i}\right)\right\|\right\|_{\infty,[x_{0},b]}\left(g\left(b\right)-g\left(x_{0}\right)\right)^{(n+1)\alpha}\left(\int_{x_{0}}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\left\|f_{j}\left(x\right)\right\|\right)dx\right)\right]\right].$$

Proof. By Theorem 4.1. □

We continue with

Corollary 5.2. *All as in Theorem 4.7, $r=2$, $n=1$, $N_{1}\left(f_{1},f_{2}\right)$ as in (53); A is a commutative Banach algebra. Then*

$$\left\|\Phi\left(f_{1},f_{2}\right)\right\|\leq\frac{\left(b-a\right)\left(g\left(b\right)-g\left(a\right)\right)^{2\alpha}N_{1}\left(f_{1},f_{2}\right)}{\Gamma\left(2\alpha+1\right)}\left(\int_{a}^{b}\left(\left\|f_{1}\left(x\right)\right\|+\left\|f_{2}\left(x\right)\right\|\right)dx\right). \tag{73}$$

Proof. By Theorem 4.7. □

We finish with

Corollary 5.3. *(to Corollary 5.1) All as in Corollary 5.1, for $g\left(t\right)=e^{t}$. Then*

$$\left\|\Omega\left(f_{1},f_{2}\right)\left(x_{0}\right)\right\|\leq\frac{1}{\Gamma\left(\left(n+1\right)\alpha+1\right)}\sum_{i=1}^{2}\left[\left[\left\|\left\|\left(D_{x_{0}-;e^{t}}^{(n+1)\alpha}f_{i}\right)\right\|\right\|_{\infty,[a,x_{0}]}\right.\right.$$
$$\left.\left(e^{x_{0}}-e^{a}\right)^{(n+1)\alpha}\left(\int_{a}^{x_{0}}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\left\|f_{j}\left(x\right)\right\|\right)dx\right)\right]+$$
$$\left.\left.\left[\left\|\left\|\left(D_{x_{0}+;e^{t}}^{(n+1)\alpha}f_{i}\right)\right\|\right\|_{\infty,[x_{0},b]}\left(e^{b}-e^{x_{0}}\right)^{(n+1)\alpha}\left(\int_{x_{0}}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{2}\left\|f_{j}\left(x\right)\right\|\right)dx\right)\right]\right]. \tag{74}$$

Proof. By Corollary 5.1. □

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