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GENERALIZED CANAVATI TYPE g-FRACTIONAL IYENGAR AND OSTROWSKI TYPE INEQUALITIES

GEORGE A. ANASTASSIOU

ABSTRACT. We present here generalized Canavati type g-fractional Iyengar and Ostrowski type inequalities. Our inequalities are with respect to all L_p norms: $1 \leq p \leq \infty$. We finish with applications.

1. BACKGROUND - I

We are motivated by the following famous Iyengar inequality (1938), [7].

Theorem 1.1. *Let f be a differentiable function on $[a, b]$ and $|f'(x)| \leq M$. Then*

$$\left| \int_a^b f(x)\, dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M}. \tag{1}$$

We need the following fractional calculus background:

Let $\alpha > 0$, $m = [\alpha]$, ($[\cdot]$ is the integral part), $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a, b])$, $[a, b] \subset \mathbb{R}$, $x \in [a, b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral ([1], p. 24)

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{2}$$

$a \leq x \leq b$. We define the subspace $C_{a+}^\alpha([a, b])$ of $C^m([a, b])$:

$$C_{a+}^\alpha([a, b]) = \left\{ f \in C^m([a, b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a, b]) \right\}. \tag{3}$$

For $f \in C_{a+}^\alpha([a, b])$, we define the left generalized α-fractional derivative of f over $[a, b]$ as

$$D_{a+}^\alpha f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \tag{4}$$

see [1], p. 24. Canavati first in [6] introduced the above over $[0, 1]$.

We have that $D_{a+}^n f = f^{(n)}$; $n \in \mathbb{N}$.

Notice that $D_{a+}^\alpha f \in C([a, b])$.

Furthermore we need:

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Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{5}$$

$x \in [a,b]$, see [2]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) = \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{6}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{7}$$

see [2]. We set $D_{b-}^0 f = f$. We have $D_{b-}^n f = (-1)^n f^{(n)}$; $n \in \mathbb{N}$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [5] we have the following Canavati fractional Iyengar type inequalities:

Theorem 1.2. *Let $\nu \geq 1$, $n = [\nu]$ and $f \in C_{a+}^{\nu}([a,b]) \cap C_{b-}^{\nu}([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \left[(t-a)^{\nu+1} + (b-t)^{\nu+1} \right], \tag{8}$$

$\forall\, t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (8) is minimized, and we get:

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{9}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left| \int_a^b f(x)\, dx \right| \leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{10}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \left\| D_{a+}^{\nu} f \right\|_{\infty,([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{\infty,([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{11}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (11) we get:

$$\left| \int_a^b f(x)\, dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{\infty,([a,b])},\left\|D_{b-}^{\nu}f\right\|_{\infty,([a,b])}\right\}}{\Gamma(\nu+2)}\left(\frac{b-a}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \quad (12)$$

$j=0,1,2,...,N,$

vi) when $N=2$ and $j=1$, (12) turns to

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{\infty,([a,b])},\left\|D_{b-}^{\nu}f\right\|_{\infty,([a,b])}\right\}}{\Gamma(\nu+2)}\frac{(b-a)^{\nu+1}}{2^{\nu}}. \quad (13)$$

We continue with L_1 estimates:

Theorem 1.3. *([5]) Let $\nu\geq 1$, $n=[\nu]$, and $f\in C_{a+}^{\nu}([a,b])\cap C_{b-}^{\nu}([a,b])$. Then*

i)

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left[(t-a)^{\nu}+(b-t)^{\nu}\right], \quad (14)$$

$\forall\, t\in[a,b]$,

ii) when $\nu=1$, from (14), we have

$$\left|\int_a^b f(x)\,dx-[f(a)(t-a)+f(b)(b-t)]\right|\leq$$

$$\|f'\|_{L_1([a,b])}(b-a),\quad \forall\, t\in[a,b], \quad (15)$$

iii) from (15), we obtain ($\nu=1$ case)

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq\|f'\|_{L_1([a,b])}(b-a), \quad (16)$$

iv) at $t=\frac{a+b}{2}$, $\nu>1$, the right hand side of (14) is minimized, and we get:

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \quad (17)$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$, for all $k=0,1,...,n-1$; $\nu>1$, from (17), we obtain

$$\left|\int_a^b f(x)\,dx\right|\leq\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \quad (18)$$

which is a sharp inequality,

vi) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, it holds

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right|$$

$$\leq \frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{19}$$

vii) if $f^{(k)}(a)=f^{(k)}(b)=0$, $k=1,...,n-1$, *from (19) we get:*

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{N}\right)\left[jf(a)+(N-j)f(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{20}$$

$j=0,1,2,...,N$,

viii) when $N=2$ *and* $j=1$, *(20) turns to*

$$\left|\int_a^b f(x)\,dx-\frac{(b-a)}{2}\left(f(a)+f(b)\right)\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{21}$$

We continue with L_p estimates:

Theorem 1.4. *([5]) Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\nu\geq 1$, $n=[\nu]$; $f\in C_{a+}^{\nu}([a,b])\cap C_{b-}^{\nu}([a,b])$. *Then*

i)

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left[(t-a)^{\nu+\frac{1}{p}}+(b-t)^{\nu+\frac{1}{p}}\right], \tag{22}$$

$\forall\, t\in[a,b]$,

ii) at $t=\frac{a+b}{2}$, *the right hand side of (22) is minimized, and we get:*

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{23}$$

iii) if $f^{(k)}(a)=f^{(k)}(b)=0$, *for all* $k=0,1,...,n-1$, *we obtain*

$$\left|\int_a^b f(x)\,dx\right|\leq\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{24}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, *it holds*

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right|$$

$$\leq \frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_q([a,b])}, \left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{25}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (25) we get:

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right)[jf(a) + (N-j) f(b)]\right| \leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_q([a,b])}, \left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{26}$$

for $j = 0, 1, 2, ..., N$,

vi) when $N = 2$ and $j = 1$, (26) turns to

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b))\right| \leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\nu}f\right\|_{L_q([a,b])}, \left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}. \tag{27}$$

Next we follow [4].

Let $g : [a, b] \to \mathbb{R}$ be a strictly increasing function. Let $f \in C^n([a, b])$, $n \in \mathbb{N}$. Assume that $g \in C^1([a, b])$, and $g^{-1} \in C^n([a, b])$. Call $l := f \circ g^{-1} : [g(a), g(b)] \to \mathbb{R}$. It is clear that $l, l', ..., l^{(n)}$ are continuous functions from $[g(a), g(b)]$ into $f([a, b]) \subseteq \mathbb{R}$.

Let $\nu \geq 1$ such that $[\nu] = n$, $n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0 < \nu < 1$, $[\nu] = 0$. Next we follow [1], pp. 7-9.

I) Let $h \in C([g(a), g(b)])$, we define the left Riemann-Liouville fractional integral as

$$(J_{\nu}^{z_0} h)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^{z} (z-t)^{\nu-1} h(t)\,dt, \tag{28}$$

for $g(a) \leq z_0 \leq z \leq g(b)$, where Γ is the gamma function; $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$.

We set $J_0^{z_0} h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)}^{\nu}([g(a), g(b)])$ of $C^{[\nu]}([g(a), g(b)])$, where $x_0 \in [a, b]$:

$$C_{g(x_0)}^{\nu}([g(a), g(b)]) := \left\{h \in C^{[\nu]}([g(a), g(b)]) : J_{1-\alpha}^{g(x_0)} h^{([\nu])} \in C^1([g(x_0), g(b)])\right\}. \tag{29}$$

So let $h \in C_{g(x_0)}^{\nu}([g(a), g(b)])$; we define the left g-generalized fractional derivative of h of order ν, of Canavati type, over $[g(x_0), g(b)]$ as

$$D_{g(x_0)}^{\nu} h := \left(J_{1-\alpha}^{g(x_0)} h^{([\nu])}\right)'. \tag{30}$$

Clearly, for $h \in C_{g(x_0)}^{\nu}([g(a), g(b)])$, there exists

$$\left(D_{g(x_0)}^{\nu} h\right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} h^{([\nu])}(t)\,dt, \tag{31}$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)])$ we have that

$$\left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \quad (32)$$

for all $g(x_0) \le z \le g(b)$. We have $D^{n}_{g(x_0)}\left(f \circ g^{-1}\right) = \left(f \circ g^{-1}\right)^{(n)}$ and $D^{0}_{g(x_0)}\left(f \circ g^{-1}\right) = f \circ g^{-1}$.

II) Next we follow [3], pp. 345-348.

Let $h \in C([g(a), g(b)])$, we define the right Riemann-Liouville fractional integral as

$$\left(J^{\nu}_{z_0-}h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z}^{z_0} (t-z)^{\nu-1} h(t)\, dt, \quad (33)$$

for $g(a) \le z \le z_0 \le g(b)$. We set $J^{0}_{z_0-}h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C^{\nu}_{g(x_0)-}([g(a), g(b)])$ of $C^{[\nu]}([g(a), g(b)])$, where $x_0 \in [a, b]$:

$$C^{\nu}_{g(x_0)-}([g(a), g(b)]) := \left\{h \in C^{[\nu]}([g(a), g(b)]) : J^{1-\alpha}_{g(x_0)-}h^{([\nu])} \in C^{1}([g(a), g(x_0)])\right\}. \quad (34)$$

So let $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)])$; we define the right g-generalized fractional derivative of h of order ν, of Canavati type, over $[g(a), g(x_0)]$ as

$$D^{\nu}_{g(x_0)-}h := (-1)^{n-1}\left(J^{1-\alpha}_{g(x_0)-}h^{([\nu])}\right)'. \quad (35)$$

Clearly, for $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)])$, there exists

$$\left(D^{\nu}_{g(x_0)-}h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t-z)^{-\alpha} h^{([\nu])}(t)\, dt, \quad (36)$$

for all $g(a) \le z \le g(x_0) \le g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)])$ we have that

$$\left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t-z)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \quad (37)$$

for all $g(a) \le z \le g(x_0) \le g(b)$.

We get that

$$\left(D^{n}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = (-1)^{n}\left(f \circ g^{-1}\right)^{(n)}(z) \quad (38)$$

and $\left(D^{0}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \left(f \circ g^{-1}\right)(z)$, all $z \in [g(a), g(x_0)]$.

Let g be strictly increasing and continuous over $[a, b]$, and $f \in C([a, b])$. We have that

$$\int_{a}^{b} f(x)\, dg(x) = \int_{a}^{b} \left(f \circ g^{-1}\right)(g(x))\, dg(x) = \int_{g(a)}^{g(b)} \left(f \circ g^{-1}\right)(z)\, dz. \quad (39)$$

Here it is $f \circ g^{-1} \in C([g(a), g(b)])$.

2. MAIN RESULTS - I

Next we present generalized g-fractional Iyengar type inequalities:

Theorem 2.1. *Let $\nu \ge 1$ such that $[\nu] = n \in \mathbb{N}$, and $g : [a, b] \to \mathbb{R}$ be a strictly increasing function. Assume that $f \in C^{n}([a, b])$, $g \in C^{1}([a, b])$, $g^{-1} \in C^{n}([g(a), g(b)])$.*

Assume further that $f \circ g^{-1} \in C_{g(a)}^{\nu}([g(a), g(b)]) \cap C_{g(b)-}^{\nu}([g(a), g(b)])$. *Set*

$$M_1(f,g) := \max\left\{ \left\| D_{g(a)}^{\nu}\left(f \circ g^{-1}\right)\right\|_{\infty,[g(a),g(b)]}, \left\| D_{g(b)-}^{\nu}\left(f \circ g^{-1}\right)\right\|_{\infty,[g(a),g(b)]} \right\}. \tag{40}$$

Then

i)

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(f \circ g^{-1}\right)^{(k)}(g(a))(z-g(a))^{k+1} + (-1)^k \left(f \circ g^{-1}\right)^{(k)}(g(b))(g(b)-z)^{k+1} \right] \right| \leq \frac{M_1(f,g)}{\Gamma(\nu+2)}\left[(z-g(a))^{\nu+1} + (g(b)-z)^{\nu+1} \right], \tag{41}$$

$\forall\, z \in [g(a), g(b)]$,

ii) at $z = \frac{g(a)+g(b)}{2}$, *the right hand side of (41) is minimized, and we get:*

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(g(b)-g(a))^{k+1}}{2^{k+1}} \left[\left(f \circ g^{-1}\right)^{(k)}(g(a)) + (-1)^k \left(f \circ g^{-1}\right)^{(k)}(g(b)) \right] \right| \leq \frac{M_1(f,g)}{\Gamma(\nu+2)} \frac{(g(b)-g(a))^{\nu+1}}{2^{\nu}}, \tag{42}$$

iii) if $\left(f \circ g^{-1}\right)^{(k)}(g(a)) = \left(f \circ g^{-1}\right)^{(k)}(g(b)) = 0$, *for all* $k = 0, 1, ..., n-1$, *we obtain*

$$\left| \int_a^b f(x)\, dg(x) \right| \leq \frac{M_1(f,g)}{\Gamma(\nu+2)} \frac{(g(b)-g(a))^{\nu+1}}{2^{\nu}}, \tag{43}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(x)\, dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{g(b)-g(a)}{N} \right)^{k+1} \left[j^{k+1} \left(f \circ g^{-1}\right)^{(k)}(g(a)) + (-1)^k (N-j)^{k+1} \left(f \circ g^{-1}\right)^{(k)}(g(b)) \right] \right| \leq \frac{M_1(f,g)}{\Gamma(\nu+2)} \left(\frac{g(b)-g(a)}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{44}$$

v) if $\left(f \circ g^{-1}\right)^{(k)}(g(a)) = \left(f \circ g^{-1}\right)^{(k)}(g(b)) = 0$, $k = 1, ..., n-1$, *from (44) we get:*

$$\left| \int_a^b f(x)\, dg(x) - \left(\frac{g(b)-g(a)}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq \frac{M_1(f,g)}{\Gamma(\nu+2)} \left(\frac{g(b)-g(a)}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{45}$$

$j = 0, 1, 2, ..., N$,

vi) when $N = 2$ and $j = 1$, (45) turns to

$$\left| \int_a^b f(x)\,dg(x) - \left(\frac{g(b) - g(a)}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{M_1(f,g)}{\Gamma(\nu+2)} \frac{(g(b) - g(a))^{\nu+1}}{2^\nu}. \tag{46}$$

Proof. Apply Theorem 1.2 for $f \circ g^{-1}$ over $[g(a), g(b)]$ and take into account (39). □

Next come L_1 estimates:

Theorem 2.2. *All as in Theorem 2.1. Set*

$$M_2(f,g) := \max \left\{ \left\| D^\nu_{g(a)} \left(f \circ g^{-1} \right) \right\|_{L_1[g(a),g(b)]}, \left\| D^\nu_{g(b)-} \left(f \circ g^{-1} \right) \right\|_{L_1[g(a),g(b)]} \right\}. \tag{47}$$

Then

i)

$$\left| \int_a^b f(x)\,dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(f \circ g^{-1} \right)^{(k)} (g(a)) (z - g(a))^{k+1} \right.\right.$$

$$\left.\left. + (-1)^k \left(f \circ g^{-1} \right)^{(k)} (g(b)) (g(b) - z)^{k+1} \right] \right| \leq$$

$$\frac{M_2(f,g)}{\Gamma(\nu+1)} \left[(z - g(a))^\nu + (g(b) - z)^\nu \right], \tag{48}$$

$\forall\, z \in [g(a), g(b)]$,

ii) when $\nu = 1$, from (48), we have

$$\left| \int_a^b f(x)\,dg(x) - \left[f(a)(z - g(a)) + f(b)(g(b) - z) \right] \right| \leq$$

$$\left\| \left(f \circ g^{-1} \right)' \right\|_{L_1([g(a),g(b)])} (g(b) - g(a)), \tag{49}$$

$\forall\, z \in [g(a), g(b)]$,

iii) from (49) we obtain ($\nu = 1$ case)

$$\left| \int_a^b f(x)\,dg(x) - \left(\frac{g(b) - g(a)}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\left\| \left(f \circ g^{-1} \right)' \right\|_{L_1([g(a),g(b)])} (g(b) - g(a)), \tag{50}$$

iv) at $z = \frac{g(a)+g(b)}{2}$, $\nu > 1$, the right hand side of (48) is minimized, and we get:

$$\left| \int_a^b f(x)\,dg(x) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(g(b) - g(a))^{k+1}}{2^{k+1}} \right.$$

$$\left. \left[\left(f \circ g^{-1} \right)^{(k)} (g(a)) + (-1)^k \left(f \circ g^{-1} \right)^{(k)} (g(b)) \right] \right| \leq$$

$$\frac{M_2(f,g)}{\Gamma(\nu+1)} \frac{(g(b) - g(a))^\nu}{2^{\nu-1}}, \tag{51}$$

v) if $\left(f\circ g^{-1}\right)^{(k)}(g(a))=\left(f\circ g^{-1}\right)^{(k)}(g(b))=0$, *for all* $k=0,1,...,n-1$; $\nu>1$, *from (51), we obtain*

$$\left|\int_a^b f(x)\,dg(x)\right|\leq\frac{M_2(f,g)}{\Gamma(\nu+1)}\frac{(g(b)-g(a))^{\nu}}{2^{\nu-1}},\tag{52}$$

which is a sharp inequality,

vi) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, *it holds*

$$\left|\int_a^b f(x)\,dg(x)-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{g(b)-g(a)}{N}\right)^{k+1}\right.$$
$$\left.\left[j^{k+1}\left(f\circ g^{-1}\right)^{(k)}(g(a))+(-1)^k(N-j)^{k+1}\left(f\circ g^{-1}\right)^{(k)}(g(b))\right]\right|$$
$$\leq\frac{M_2(f,g)}{\Gamma(\nu+1)}\left(\frac{g(b)-g(a)}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right],\tag{53}$$

vii) if $\left(f\circ g^{-1}\right)^{(k)}(g(a))=\left(f\circ g^{-1}\right)^{(k)}(g(b))=0$, $k=1,...,n-1$, *from (53) we get:*

$$\left|\int_a^b f(x)\,dg(x)-\left(\frac{g(b)-g(a)}{N}\right)\left[jf(a)+(N-j)f(b)\right]\right|\leq$$
$$\frac{M_2(f,g)}{\Gamma(\nu+1)}\left(\frac{g(b)-g(a)}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right],\tag{54}$$

$j=0,1,2,...,N$,

viii) when $N=2$ *and* $j=1$, *(54) turns to*

$$\left|\int_a^b f(x)\,dg(x)-\left(\frac{g(b)-g(a)}{2}\right)(f(a)+f(b))\right|\leq$$
$$\frac{M_2(f,g)}{\Gamma(\nu+1)}\frac{(g(b)-g(a))^{\nu}}{2^{\nu-1}}.\tag{55}$$

Proof. Application of Theorem 1.3 for $f\circ g^{-1}$ over $[g(a),g(b)]$ and take into account (39). □

We continue with L_p estimates:

Theorem 2.3. *All as in Theorem 2.1. Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Set*

$$M_3(f,g):=\max\left\{\left\|D_{g(a)}^{\nu}\left(f\circ g^{-1}\right)\right\|_{L_q[g(a),g(b)]},\left\|D_{g(b)-}^{\nu}\left(f\circ g^{-1}\right)\right\|_{L_q[g(a),g(b)]}\right\}.\tag{56}$$

Then

i)

$$\left|\int_a^b f(x)\,dg(x)-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\left(f\circ g^{-1}\right)^{(k)}(g(a))(z-g(a))^{k+1}\right.\right.$$
$$\left.\left.+(-1)^k\left(f\circ g^{-1}\right)^{(k)}(g(b))(g(b)-z)^{k+1}\right]\right|\leq$$
$$\frac{M_3(f,g)}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left[(z-g(a))^{\nu+\frac{1}{p}}+(g(b)-z)^{\nu+\frac{1}{p}}\right],\tag{57}$$

$\forall\, z\in[g(a),g(b)]$,

ii) at $z=\frac{g(a)+g(b)}{2}$*, the right hand side of (57) is minimized, and we get:*

$$\left|\int_a^b f(x)\,dg(x)-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(g(b)-g(a))^{k+1}}{2^{k+1}}\right.$$
$$\left.\left[\left(f\circ g^{-1}\right)^{(k)}(g(a))+(-1)^k\left(f\circ g^{-1}\right)^{(k)}(g(b))\right]\right|\leq$$
$$\frac{M_3(f,g)}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(g(b)-g(a))^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{58}$$

iii) if $\left(f\circ g^{-1}\right)^{(k)}(g(a))=\left(f\circ g^{-1}\right)^{(k)}(g(b))=0$*, for all* $k=0,1,...,n-1$*, we obtain*

$$\left|\int_a^b f(x)\,dg(x)\right|\leq\frac{M_3(f,g)}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(g(b)-g(a))^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{59}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$*, it holds*

$$\left|\int_a^b f(x)\,dg(x)-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{g(b)-g(a)}{N}\right)^{k+1}\right.$$
$$\left.\left[j^{k+1}\left(f\circ g^{-1}\right)^{(k)}(g(a))+(-1)^k(N-j)^{k+1}\left(f\circ g^{-1}\right)^{(k)}(g(b))\right]\right|\leq$$
$$\frac{M_3(f,g)}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\frac{g(b)-g(a)}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{60}$$

v) if $\left(f\circ g^{-1}\right)^{(k)}(g(a))=\left(f\circ g^{-1}\right)^{(k)}(g(b))=0$*,* $k=1,...,n-1$*, from (60) we get:*

$$\left|\int_a^b f(x)\,dg(x)-\left(\frac{g(b)-g(a)}{N}\right)[jf(a)+(N-j)f(b)]\right|\leq$$
$$\frac{M_3(f,g)}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\frac{g(b)-g(a)}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{61}$$

$j=0,1,2,...,N$,

vi) when $N=2$ *and* $j=1$*, (61) turns to*

$$\left|\int_a^b f(x)\,dg(x)-\left(\frac{g(b)-g(a)}{2}\right)(f(a)+f(b))\right|\leq$$
$$\frac{M_3(f,g)}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(g(b)-g(a))^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}. \tag{62}$$

Proof. Application of Theorem 1.4 to $f\circ g^{-1}$ over $[g(a),g(b)]$ and using (39). ☐

3. Background - II

In 1938, A. Ostsrowski [8] proved the following important inequality.

Theorem 3.1. *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *whose derivative* $f':(a,b)\to\mathbb{R}$ *is bounded on* (a,b)*, i.e.,* $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<+\infty$.

Then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \cdot (b-a)\left\|f'\right\|_\infty, \tag{63}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

We mention and need the following left generalized g-fractional, of Canavati type, Taylor's formula:

Theorem 3.2. *([4]) Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)])$, where $x_0 \in [a,b]$ is fixed, $\nu \geq 1$. Then*

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!}(g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)}\int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1}\left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(t)\,dt, \text{ all } x \in [a,b] : x \geq x_0. \tag{64}$$

We also mention and need the following right generalized g-fractional, of Canavati type, Taylor's formula:

Theorem 3.3. *([4]) Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)])$, where $x_0 \in [a,b]$ is fixed, $\nu \geq 1$. Then*

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!}(g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)}\int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1}\left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\,dt, \text{ all } a \leq x \leq x_0. \tag{65}$$

4. MAIN RESULTS - II

Next we present generalized g-fractional Ostrowski type inequalities:

Theorem 4.1. *Let $g : [a,b] \to \mathbb{R}$ be a strictly increasing function, $\nu \geq 1$, $[\nu] = n \in \mathbb{N}$, $f \in C^n([a,b])$. Assume that $g \in C^1([a,b])$, and $g^{-1} \in C^n([g(a), g(b)])$. For $x_0 \in [a,b]$, assume that $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)])$ and $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)])$. Furthermore assume that $\left(f \circ g^{-1}\right)^{(k)}(g(x_0)) = 0$, all $k = 1, \ldots, n-1$. Then*

$$\left|\frac{1}{(g(b)-g(a))}\int_a^b f(x)\,dg(x) - f(x_0)\right| \leq$$

$$\frac{1}{(g(b)-g(a))}\left\{\left\|D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right\|_{\infty,[g(a),g(x_0)]}(g(x_0)-g(a))^{\nu+1} + \right.$$

$$\left.\left\|D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^{\nu+1}\right\} \leq \tag{66}$$

$$\frac{1}{\Gamma(\nu+2)}\max\left\{\left\|D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right\|_{\infty,[g(a),g(x_0)]},\right.$$

$$\left.\left\|D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right\|_{\infty,[g(x_0),g(b)]}\right\}(g(b)-g(a))^{\nu}.$$

Proof. By Theorem 3.2, when $\left(f\circ g^{-1}\right)^{(k)}\left(g\left(x_0\right)\right)=0$, for $k=1,...,n-1$, we get

$$f\left(x\right)-f\left(x_0\right)=\frac{1}{\Gamma\left(\nu\right)}\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-t\right)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f\circ g^{-1}\right)\right)\left(t\right)dt, \tag{67}$$

$\forall\, x\in\left[x_0,b\right].$

By Theorem 3.3, when $\left(f\circ g^{-1}\right)^{(k)}\left(g\left(x_0\right)\right)=0$, for $k=1,...,n-1$, we get

$$f\left(x\right)-f\left(x_0\right)=\frac{1}{\Gamma\left(\nu\right)}\int_{g(x)}^{g(x_0)}\left(t-g\left(x\right)\right)^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f\circ g^{-1}\right)\right)\left(t\right)dt, \tag{68}$$

$\forall\, x\in\left[a,x_0\right].$

Hence

$$\begin{aligned}\left|f\left(x\right)-f\left(x_0\right)\right| &\leq \frac{1}{\Gamma\left(\nu\right)}\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-t\right)^{\nu-1}\left|\left(D_{g(x_0)}^{\nu}\left(f\circ g^{-1}\right)\right)\left(t\right)\right|dt\\ &\leq \frac{1}{\Gamma\left(\nu\right)}\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-t\right)^{\nu-1}dt\right)\left\|D_{g(x_0)}^{\nu}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(x_0),g(b)]}\\ &= \frac{1}{\Gamma\left(\nu\right)}\frac{\left(g\left(x\right)-g\left(x_0\right)\right)^{\nu}}{\nu}\left\|D_{g(x_0)}^{\nu}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(x_0),g(b)]}\\ &= \frac{\left(g\left(x\right)-g\left(x_0\right)\right)^{\nu}}{\Gamma\left(\nu+1\right)}\left\|D_{g(x_0)}^{\nu}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(x_0),g(b)]},\end{aligned} \tag{69}$$

$\forall\, x\in\left[x_0,b\right].$

That is

$$\left|f\left(x\right)-f\left(x_0\right)\right|\leq\frac{\left\|D_{g(x_0)}^{\nu}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(x_0),g(b)]}}{\Gamma\left(\nu+1\right)}\left(g\left(x\right)-g\left(x_0\right)\right)^{\nu}, \tag{70}$$

$\forall\, x\in\left[x_0,b\right].$

Similarly, it holds

$$\begin{aligned}\left|f\left(x\right)-f\left(x_0\right)\right| &\leq \frac{1}{\Gamma\left(\nu\right)}\int_{g(x)}^{g(x_0)}\left(t-g\left(x\right)\right)^{\nu-1}\left|\left(D_{g(x_0)-}^{\nu}\left(f\circ g^{-1}\right)\right)\left(t\right)\right|dt\\ &\leq \frac{1}{\Gamma\left(\nu\right)}\left(\int_{g(x)}^{g(x_0)}\left(t-g\left(x\right)\right)^{\nu-1}dt\right)\left\|D_{g(x_0)-}^{\nu}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(a),g(x_0)]}\\ &= \frac{1}{\Gamma\left(\nu+1\right)}\left(g\left(x_0\right)-g\left(x\right)\right)^{\nu}\left\|D_{g(x_0)-}^{\nu}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(a),g(x_0)]},\end{aligned} \tag{71}$$

$\forall\, x\in\left[a,x_0\right].$

That is

$$\left|f\left(x\right)-f\left(x_0\right)\right|\leq\frac{\left\|D_{g(x_0)-}^{\nu}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(a),g(x_0)]}}{\Gamma\left(\nu+1\right)}\left(g\left(x_0\right)-g\left(x\right)\right)^{\nu}, \tag{72}$$

$\forall\, x\in\left[a,x_0\right].$

We observe that

$$\left|\frac{1}{\left(g\left(b\right)-g\left(a\right)\right)}\int_a^b f\left(x\right)g'\left(x\right)dx-f\left(x_0\right)\right|=$$

$$\left|\frac{1}{(g(b)-g(a))}\int_a^b f(x)\,g'(x)\,dx - f(x_0)(g(b)-g(a))\right| =$$

$$\left|\frac{1}{(g(b)-g(a))}\left[\int_a^b f(x)\,g'(x)\,dx - \int_a^b f(x_0)\,g'(x)\,dx\right]\right| =$$

$$\left|\frac{1}{(g(b)-g(a))}\left[\int_a^b (f(x)-f(x_0))\,g'(x)\,dx\right]\right| \leq \tag{73}$$

$$\frac{1}{(g(b)-g(a))}\int_a^b |f(x)-f(x_0)|\,g'(x)\,dx =$$

$$\frac{1}{(g(b)-g(a))}\left\{\int_a^{x_0} |f(x)-f(x_0)|\,g'(x)\,dx + \int_{x_0}^b |f(x)-f(x_0)|\,g'(x)\,dx\right\}$$

$$\overset{\text{(by (70), (72))}}{\leq} \frac{1}{(g(b)-g(a))}\left\{\frac{\left\|D^{\nu}_{g(x_0)-}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(a),g(x_0)]}}{\Gamma(\nu+1)}\right.$$

$$\int_a^{x_0} (g(x_0)-g(x))^{\nu}\,g'(x)\,dx +$$

$$\left.\frac{\left\|D^{\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(x_0),g(b)]}}{\Gamma(\nu+1)}\int_{x_0}^b (g(x)-g(x_0))^{\nu}\,g'(x)\,dx\right\} = \tag{74}$$

$$\frac{1}{(g(b)-g(a))\,\Gamma(\nu+2)}\left\{\left\|D^{\nu}_{g(x_0)-}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(a),g(x_0)]}(g(x_0)-g(a))^{\nu+1}\right.$$

$$\left.+\left\|D^{\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^{\nu+1}\right\} \leq$$

$$\frac{1}{\Gamma(\nu+2)}\max\left\{\left\|D^{\nu}_{g(x_0)-}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(a),g(x_0)]},\right.$$

$$\left.\left\|D^{\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right\|_{\infty,[g(x_0),g(b)]}\right\}(g(b)-g(a))^{\nu}. \tag{75}$$

□

We continue with an L_1-estimate:

Theorem 4.2. *All as in Theorem 4.1. Then*

$$\left|\frac{1}{(g(b)-g(a))}\int_a^b f(x)\,dg(x) - f(x_0)\right| \leq$$

$$\frac{1}{(g(b)-g(a))\,\Gamma(\nu+1)}\left\{\left\|D^{\nu}_{g(x_0)-}\left(f\circ g^{-1}\right)\right\|_{L_1([g(a),g(x_0)])}(g(x_0)-g(a))^{\nu}\right.$$

$$\left.+\left\|D^{\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right\|_{L_1[g(x_0),g(b)]}(g(b)-g(x_0))^{\nu}\right\} \leq$$

$$\frac{1}{\Gamma(\nu+1)}\max\left\{\left\|D^{\nu}_{g(x_0)-}\left(f\circ g^{-1}\right)\right\|_{L_1([g(a),g(x_0)])},\right.$$

$$\left.\left\|D^{\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right\|_{L_1([g(x_0),g(b)])}\right\}(g(b)-g(a))^{\nu-1}. \tag{76}$$

Proof. Similar to the proof of Theorem 4.1. □

We continue with an L_p-estimate:

Theorem 4.3. *All as in Theorem 4.1, and let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\left| \frac{1}{(g(b) - g(a))} \int_a^b f(x)\, dg(x) - f(x_0) \right| \leq$$

$$\frac{1}{(g(b) - g(a))\, \Gamma(\nu)\, (p(\nu - 1) + 1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)}$$

$$\left\{ (g(x_0) - g(a))^{\nu + \frac{1}{p}} \left\| D^{\nu}_{g(x_0)-} \left(f \circ g^{-1}\right) \right\|_{L_q([g(a), g(x_0)])} \right.$$

$$\left. + (g(b) - g(x_0))^{\nu + \frac{1}{p}} \left\| D^{\nu}_{g(x_0)} \left(f \circ g^{-1}\right) \right\|_{L_q[g(x_0), g(b)]} \right\} \leq$$

$$\frac{1}{\Gamma(\nu)\, (p(\nu - 1) + 1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} \max \left\{ \left\| D^{\nu}_{g(x_0)-} \left(f \circ g^{-1}\right) \right\|_{L_q([g(a), g(x_0)])}, \right.$$

$$\left. \left\| D^{\nu}_{g(x_0)} \left(f \circ g^{-1}\right) \right\|_{L_q([g(x_0), g(b)])} \right\} (g(b) - g(a))^{\nu - \frac{1}{q}}. \tag{77}$$

Proof. Similar to the proof of Theorem 4.1. □

Applications follow:

Proposition 4.4. *Let* $\nu \geq 1$ *such that* $[\nu] = n \in \mathbb{N}$, *and* $f \in C^n([a, b])$, *where* $[a, b] \subset (0, +\infty)$. *Assume that* $f \circ \ln x \in C^{\nu}_{e^a}\left(\left[e^a, e^b\right]\right) \cap C^{\nu}_{e^b-}\left(\left[e^a, e^b\right]\right)$. *Set*

$$M_1(f, e^x) := \max \left\{ \left\| D^{\nu}_{e^a} (f \circ \ln x) \right\|_{\infty, [e^a, e^b]}, \left\| D^{\nu}_{e^b-} (f \circ \ln x) \right\|_{\infty, [e^a, e^b]} \right\}. \tag{78}$$

Then

i)

$$\left| \int_a^b f(x)\, e^x dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(f \circ \ln x)^{(k)} (e^a) (z - e^a)^{k+1} \right. \right.$$

$$\left. \left. + (-1)^k (f \circ \ln x)^{(k)} \left(e^b\right) \left(e^b - z\right)^{k+1} \right] \right| \leq$$

$$\frac{M_1(f, e^x)}{\Gamma(\nu + 2)} \left[(z - e^a)^{\nu+1} + \left(e^b - z\right)^{\nu+1} \right], \tag{79}$$

$\forall\, z \in \left[e^a, e^b\right]$,

ii) at $z = \frac{e^a + e^b}{2}$, *the right hand side of (79) is minimized, and we get:*

$$\left| \int_a^b f(x)\, e^x dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{\left(e^b - e^a\right)^{k+1}}{2^{k+1}} \right.$$

$$\left. \left[(f \circ \ln x)^{(k)} (e^a) + (-1)^k (f \circ \ln x)^{(k)} \left(e^b\right) \right] \right| \leq$$

$$\frac{M_1(f, e^x)}{\Gamma(\nu + 2)} \frac{\left(e^b - e^a\right)^{\nu+1}}{2^{\nu}}, \tag{80}$$

iii) if $(f\circ\ln x)^{(k)}(e^a)=(f\circ\ln x)^{(k)}(e^b)=0$, *for all* $k=0,1,...,n-1$, *we obtain*

$$\left|\int_a^b f(x)e^x dx\right|\leq\frac{M_1(f,e^x)}{\Gamma(\nu+2)}\frac{(e^b-e^a)^{\nu+1}}{2^\nu},\tag{81}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, *it holds*

$$\left|\int_a^b f(x)e^x dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{e^b-e^a}{N}\right)^{k+1}\right.$$
$$\left.\left[j^{k+1}(f\circ\ln x)^{(k)}(e^a)+(-1)^k(N-j)^{k+1}(f\circ\ln x)^{(k)}(e^b)\right]\right|$$
$$\leq\frac{M_1(f,e^x)}{\Gamma(\nu+2)}\left(\frac{e^b-e^a}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right],\tag{82}$$

v) if $(f\circ\ln x)^{(k)}(e^a)=(f\circ\ln x)^{(k)}(e^b)=0$, $k=1,...,n-1$, *from (82) we get:*

$$\left|\int_a^b f(x)e^x dx-\left(\frac{e^b-e^a}{N}\right)[jf(a)+(N-j)f(b)]\right|\leq$$
$$\frac{M_1(f,e^x)}{\Gamma(\nu+2)}\left(\frac{e^b-e^a}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right],\tag{83}$$

$j=0,1,2,...,N$,

vi) when $N=2$ *and* $j=1$, *(83) turns to*

$$\left|\int_a^b f(x)e^x dx-\left(\frac{e^b-e^a}{2}\right)(f(a)+f(b))\right|\leq$$
$$\frac{M_1(f,e^x)}{\Gamma(\nu+2)}\frac{(e^b-e^a)^{\nu+1}}{2^\nu}.\tag{84}$$

Proof. By application of Theorem 2.1 for $g(x)=e^x$. □

We finish with

Proposition 4.5. *Let* $\nu\geq1$, $[\nu]=n\in\mathbb{N}$, $f\in C^n([a,b])$, $[a,b]\subset(0,+\infty)$. *For* $x_0\in[a,b]$, *assume that* $f\circ e^x\in C^\nu_{\ln x_0}([\ln a,\ln b])$, *and* $f\circ e^x\in C^\nu_{\ln x_0-}([\ln a,\ln b])$. *Furthermore assume that* $(f\circ e^x)^{(k)}(\ln x_0)=0$, *all* $k=1,...,n-1$. *Then*

$$\left|\frac{1}{\left(\ln\frac{b}{a}\right)}\int_a^b\frac{f(x)}{x}dx-f(x_0)\right|\leq$$
$$\frac{1}{\left(\ln\frac{b}{a}\right)\Gamma(\nu+2)}\left\{\left\|D^\nu_{\ln x_0-}(f\circ e^x)\right\|_{\infty,[\ln a,\ln x_0]}\left(\ln\frac{x_0}{a}\right)^{\nu+1}+\right.$$
$$\left.\left\|D^\nu_{\ln x_0}(f\circ e^x)\right\|_{\infty,[\ln x_0,\ln b]}\left(\ln\frac{b}{x_0}\right)^{\nu+1}\right\}\leq\tag{85}$$
$$\frac{1}{\Gamma(\nu+2)}\max\left\{\left\|D^\nu_{\ln x_0-}(f\circ e^x)\right\|_{\infty,[\ln a,\ln x_0]},\right.$$
$$\left.\left\|D^\nu_{\ln x_0}(f\circ e^x)\right\|_{\infty,[\ln x_0,\ln b]}\right\}\left(\ln\frac{b}{a}\right)^\nu.$$

Proof. By Theorem 4.1, for $g(x)=\ln x$. □

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GEORGE A. ANASTASSIOU

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF MEMPHIS, MEMPHIS, TN 38152, U.S.A.

Email address: ganastss@memphis.edu