ON \textit{m}\text{-EXPANSIVE AND m\text{-CONTRACTIVE TUPLE OF OPERATORS IN HILBERT SPACES}

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\textbf{Abstract.} In this paper we introduce and studied the concept of joint \textit{m}\text{-expansive and joint \textit{m}\text{-contractive tuples of commuting operators of a Hilbert space.}

1. INTRODUCTION

In this paper \(\mathcal{H}\) will denote a infinite-dimensional Hilbert space on \(\mathbb{K} = \mathbb{C}\) (the complex plane). \(\mathbb{N}\) is the set of positive integers and \(\mathbb{Z}_{+} = \mathbb{N} \cup \{0\}\). Let \(\mathcal{B}(\mathcal{H})\) be the set of bounded linear operators from \(\mathcal{H}\) into itself. An operator \(S \in \mathcal{B}(\mathcal{H})\) we denote by \(\mathcal{N}(S)\) and \(\mathcal{R}(S)\) the null space and the range of \(S\) respectively. For \(S \in \mathcal{B}(\mathcal{H})\), we set

\[\theta_m(S) := \sum_{0 \leq j \leq m} (-1)^j \binom{m}{j} S^* S^j.\]

J. Agler and M. Stankus introduced the class of \textit{m}\text{-isometry on Hilbert space \([1, 2, 3]\). An operator \(S \in \mathcal{B}(\mathcal{H})\) is said to be \textit{m}\text{-isometric operator for some integer \(m \geq 1\) if it satisfies the operator equation \(\theta_m(S) = 0\).

Notice that the defining property \(\theta_m(S) = 0\) of an \textit{m}\text{-isometric operator is equivalently formulated that

\[\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \|S^k x\|^2 = 0 \quad (\forall x \in \mathcal{H}).\]

The Concept of \textit{m}\text{-isometric operators on Hilbert and Banach spaces has attracted much attention of various authors (see \([8, 2, 10, 11, 12, 13, 17, 21, 22, 19]\). A generalization of \textit{m}\text{-isometries to \textit{m}\text{-expansive and \textit{m}\text{-contractive operators on Hilbert spaces spaces has been presented by several authors. We refer the reader to \([4, 5, 6, 7, 14, 15, 18, 20, 23]\) for recent articles about these subjects.

\textbf{Definition 1.1.} (\([14]\)) An operator \(S \in \mathcal{B}(\mathcal{H})\) is said to be

\begin{enumerate}[(i)]
  \item \textit{m}\text{-expansive \((m \geq 1)\) if } \theta_m(S) \leq 0.
\end{enumerate}
(ii) \( m \)-hyperexpensive \( (m \geq 1) \), if \( \theta_k(S) \leq 0 \) for \( k = 1, 2, \ldots, m \).

(iii) Completely hyperexpansive if \( \theta_m(S) \leq 0 \) for all \( m \).

**Definition 1.2.** (15) An operator \( S \in \mathcal{B}(\mathcal{H}) \) is said to be

(i) \( m \)-contractive \( (m \geq 1) \) if \( \theta_m(S) \geq 0 \).

(ii) \( m \)-hypercontractive \( (m \geq 1) \), if \( \theta_k(S) \geq 0 \) for \( k = 1, 2, \ldots, m \).

(iii) Completely hypercontractive if \( \theta_m(S) \geq 0 \) for all \( m \).

The study of tuple of commuting operators on Hilbert space has consider by many authors in the recent years. In [16] the authors introduced the concept of \( m \)-isomeric tuple of commuting operators as follows: Let \( S = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d \) be tuple of commuting operators. \( S \) is said to be \( m \)-isometric tuple if

\[
\sum_{0 \leq k \leq m} (-1)^k \left( \sum_{|\alpha|=k} \frac{k!}{\beta!} |S^\alpha S^\beta| \right) = 0,
\]

where \( \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_+^d \) and \( \beta! = \beta_1! \cdots \beta_d! \).

2. MAIET RESULTS

In this section, we introduce and study the concepts on \( m \)-expansive and \( m \)-contractive tuples of operators on a Hilbert space.

Let \( S = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d \) and set

\[
\Psi_l(S) = \sum_{0 \leq k \leq l} (-1)^k \left( \sum_{|\alpha|=k} \frac{k!}{\beta!} |S^\alpha S^\beta| \right),
\]

and

\[
Q_l(S, x) := \langle Sx, x \rangle = \sum_{0 \leq k \leq l} (-1)^k \binom{m}{k} \left( \sum_{|\alpha|=k} \frac{k!}{\beta!} |S^\alpha x|^2 \right).
\]

Clearly,

\[
\Psi_l(S) \geq 0 \iff Q_l(S, x) \geq 0 \quad \forall \ x \in \mathcal{H},
\]

and

\[
\Psi_l(S) \leq 0 \iff Q_l(S, x) \leq 0 \quad \forall \ x \in \mathcal{H},
\]

**Definition 2.1.** Let \( S = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d \) be a commuting tuple of operators and \( m \in \mathbb{N} \). We said that

(1) \( S \) is joint \( m \)-expansive if \( \Psi_m(S) \leq 0 \) for some integer \( m \).

(2) \( S \) is joint \( m \)-hyperexpansive tuple if \( \Psi_k(S) \leq 0 \) for each \( k = 1, 2, \ldots, m \).

(3) \( S \) is joint completely hyperexpansive tuple if \( \Psi_k(S) \leq 0 \) for all \( k \in \mathbb{N} \).

(4) \( S \) is joint \( m \)-contractive if \( \Psi_m(S) \geq 0 \) for some integer \( m \).

(5) \( S \) is joint \( m \)-hypercontractive if \( \Psi_k(S) \geq 0 \) for each \( k = 1, 2, \ldots, m \).

(6) \( S \) is joint completely hypercontractive if \( S \) is joint \( k \)-contractive for all positive integer \( k \).
When $d = 1$, Definition 2.1 coincides with Definition 1.1 and Definition 1.2.

**Remark.** Observe that

$$
\langle \Psi_l(S)x, x \rangle = \sum_{0 \leq k \leq l} (-1)^k \left( \sum_{|\beta| = k} \frac{k!}{\beta!} \|S^\beta x\|^2 \right), \quad \forall \ x \in \mathcal{H}.
$$

Then:

(1) $S$ is $m$-expansive tuple if and only if

$$
\sum_{0 \leq k \leq l} (-1)^k \left( \sum_{|\beta| = k} \frac{k!}{\beta!} \|S^\beta x\|^2 \right) \leq 0, \quad \forall \ x \in \mathcal{H},
$$

and

(2) $S$ is $m$-contractive tuple if and only if

$$
\sum_{0 \leq k \leq l} (-1)^k \left( \sum_{|\beta| = k} \frac{k!}{\beta!} \|S^\beta x\|^2 \right) \geq 0, \quad \forall \ x \in \mathcal{H}.
$$

**Remark.** (i) Let $S = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting tuple of operators. Then $S$ is a joint expansive tuple if

$$
\|x\|^2 \leq \sum_{1 \leq j \leq d} \|S_j x\|^2, \quad (\forall \ x \in \mathcal{H}) \quad (2.3)
$$

and it is a joint contractive tuple if

$$
\|x\|^2 \geq \sum_{1 \leq j \leq d} \|S_j x\|^2, \quad (\forall \ x \in \mathcal{H}). \quad (2.4)
$$

(ii) If $d = 2$, let $S = (S_1, S_2) \in \mathcal{B}(\mathcal{H})^2$ be a commuting pair of operators. Then $S$ is a joint 2-expansive pair if

$$
\|x\|^2 \leq 2(\|S_1 x\|^2 + \|S_2 x\|^2) - (\|S_1^2 x\|^p + \|S_2^2 x\|^p + 2\|S_1 S_2 x\|^p) \quad (\forall \ x \in \mathcal{H}), \quad (2.5)
$$

and it is a joint $(2, p)$-contractive pair if

$$
\|x\|^p \geq 2(\|S_1 x\|^p + \|S_2 x\|^p) - (\|S_1^2 x\|^p + \|S_2^2 x\|^p + 2\|S_1 S_2 x\|^p) \quad (\forall \ x \in \mathcal{H}). \quad (2.6)
$$

(iii) Let $S = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting tuple of operators. Then $S$ is a joint 2-expansive tuple if

$$
\|x\|^2 \leq 2 \sum_{1 \leq j \leq d} \|S_j x\|^2 - \left( \sum_{1 \leq j \leq d} \|S_j^2 x \|^2 + 2 \sum_{1 \leq j < k \leq d} \|S_j S_k x\|^2 \right) \quad \forall \ x \in \mathcal{H}, \quad (2.7)
$$

and it is a joint 2-contractive tuple if

$$
\|x\|^2 \geq 2 \sum_{1 \leq j \leq d} \|S_j x\|^2 - \left( \sum_{1 \leq j \leq d} \|S_j^2 x \|^p + 2 \sum_{1 \leq j < k \leq d} \|S_j S_k x\|^p \right) \quad \forall \ x \in \mathcal{H}. \quad (2.8)
$$

**Remark.** Since the operators $S_1, \ldots, S_d$ are commuting, every permutation of joint $m$-expansive tuple is also joint $m$-expansive tuple.

The following examples show that there exists a joint $(m, p)$-expansive (resp. joint $(m, p)$-contractive) operator which is not $(m, p)$-isometric tuple for some positive integer $m$.

**Example 2.2.** Let $\mathcal{H} = \mathbb{C}^3$ be equipped with the norm

$$
\|(x, y, z)\|_2 = \left( |x|^2 + |y|^2 + |z|^2 \right)^{\frac{1}{2}}
$$
and consider
\[
S_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3) \text{ and } S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3).
\]

Then the pair \( S = (S_1, S_2) \) is a joint 2-expansive pair on \( (\mathcal{H} = \mathbb{C}^3, \| \cdot \|_2) \).

In fact, obviously \( S_1 S_2 = S_2 S_1 \) and a simple computation shows that
\[
2 \left\| S_1 \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\|_2^p + 2 \left\| S_2 \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\|_2^2 - \left( \left\| S_1^2 \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\|_2^2 + \left\| S_2^2 \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\|_2^2 + 2 \left\| S_1 S_2 \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\|_2^2 \right) = \left( 9^2 - 1 + 3^2 \right) \left| u + v + w \right|^2
\]
\[
\geq \left\| \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right\|_2^2.
\]

However \( S \) is not a 2-isometric tuple.

**Proposition 2.1.** Let \( S = (S_1, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d \) be a tuple of commuting operators. Then for all positive integer \( m \), and \( x \in \mathcal{H} \), we have
\[
Q_{m+1}(S, x) = Q_m(S, x) - \sum_{1 \leq k \leq n} Q_m(S, S_k x). \tag{2.9}
\]

**Proof.** By taking into account Equation (2.4), a straightforward calculation shows that
\[
Q_{m+1}(S, x) = \sum_{0 \leq k \leq m+1} (-1)^k \binom{m+1}{k} \sum_{|\beta| = k} \frac{k!}{\beta!} \| S^\beta x \|^2
\]
\[
= \| x \|^2 + \sum_{1 \leq k \leq m} (-1)^k \left( \binom{m}{k} + \binom{m}{k-1} \right) \sum_{|\beta| = k} \frac{k!}{\beta!} \| S^\beta x \|^2
\]
\[
+ (-1)^{m+1} \sum_{|\beta| = m+1} \frac{(m+1)!}{\beta!} \| S^\beta x \|^2
\]
\[
= Q_m(S, x) - \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\beta| = k+1} \frac{(k+1)!}{\beta!} \| S^\beta x \|^2
\]
\[
+ (-1)^{m+1} \sum_{|\beta| = m+1} \frac{(m+1)!}{\beta!} \| S^\beta x \|^2
\]
\[
= Q_m(S, x) - \sum_{0 \leq k \leq m-1} (-1)^{m-k} \binom{m}{k} \sum_{|\beta| = k+1} \frac{k!(\beta_1 + \cdots + \beta_n)}{\beta_1! \cdots \beta_n!} \| S^\beta x \|^2
\]
\[
+ (-1)^{m+1} \sum_{|\beta| = m+1} \frac{m!(\beta_1 + \cdots + \beta_n)}{\beta_1! \cdots \beta_n!} \| S^\beta x \|^2.
\]
and so the equality (2.5) is satisfied.

\[ \Box \]

**Example 2.3.** Let \( \mathcal{H} \) be an Hilbert space and \( I_\mathcal{H} \) the identity operator. Then \((5I_\mathcal{H}, I_\mathcal{H}, I_\mathcal{H}) \in \mathcal{B}(\mathcal{H})^3 \) is a joint 2-contractive of operators which is not a 2-isometric tuple.

It is well-known that the class of \( m \)-isometric tuple is a subset of the class of \( (m + 1) \)-isometric tuple. The following example shows that the class of joint \( m \)-expansive tuple and joint \( m + 1 \)-expansive tuple are independent.

**Example 2.4.** Let \( \mathbf{S} = (I_\mathcal{H}, I_\mathcal{H}, I_\mathcal{H}) \in \mathcal{B}(\mathcal{H})^3 \). A simple computation shows that

1. \( \mathbf{S} \) is a joint 1-expansive tuple but not a joint 2-expansive tuple.
2. \( \mathbf{S} \) is a joint 2-contractive tuple but not a joint 1-contractive tuple.

The following theorem gives some sufficient conditions under which a joint \( m \)-expansive tuple of operators in Hilbert space is a joint \( m \)-hyperexpansive tuple for \( m \geq 2 \).

**Theorem 2.2.** Let \( \mathbf{S} = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d \) be a commuting tuple of operators. If \( \mathbf{S} \) is a joint \( m \)-expansive tuple and satisfies the following conditions

1. \( S_j^k \to 0 \) strongly for all \( j \in \{1, \cdots, d\} \).
2. \( S_j^k \Psi_k(S)S_j \leq S_j^{2k} \Psi_k(S)S_j^2 \) for all \( j \in \{1, \cdots, d\} \) and \( k \in \{1, \cdots, m - 1\} \),

then \( \mathbf{S} \) is a joint \( m \)-hyperexpansive tuple.

**Proof.** As \( \mathbf{S} \) is a joint \( m \)-expansive tuple, it follows that

\[
\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \left( \sum_{|\beta|=k} \frac{k!}{\beta_1! \beta_2! \cdots \beta_m!} \right) \leq 0.
\]
A computation shows that
\[
0 \geq I_H + \sum_{1 \leq k \leq m-1} (-1)^k \binom{m}{k} \left( \sum_{|\alpha|=k} \frac{k!}{\alpha!} S^\alpha S^\beta \right) + (-1)^m \sum_{|\beta|=m} \frac{m!}{\beta!} S^\beta S^\beta
\]
\[
= I_H + \sum_{1 \leq k \leq m-1} (-1)^k \binom{m-1}{k} \left( \frac{m-1}{k-1} \right) \sum_{|\alpha|=k} \frac{k!}{\alpha!} S^\alpha S^\beta + (-1)^m \sum_{|\beta|=m} \frac{m!}{\beta!} S^\beta S^\beta
\]
\[
= \sum_{0 \leq k \leq m-1} \left( -1 \right)^k \binom{m-1}{k} \sum_{|\beta|=k} \frac{k!}{\beta!} S^\beta S^\beta - \sum_{0 \leq k \leq m-2} \left( -1 \right)^k \binom{m-1}{k} \sum_{|\beta|=k+1} \frac{(k+1)!}{\beta!} S^\beta S^\beta
\]
\[
+ (-1)^m \sum_{|\beta|=m} \frac{m!}{\beta!} S^\beta S^\beta
\]
In view of the statement (2), we obtain
\[
\Psi_{m-1}(S) \leq \sum_{1 \leq j \leq d} S_j^* \left( \sum_{0 \leq k \leq m-1} \left( -1 \right)^k \binom{m-1}{k} \sum_{|\beta|=k} \frac{k!}{\beta!} S^\beta S^\beta \right) S_j
\]
\[
= \sum_{1 \leq j \leq d} S_j^* \Psi_{m-1}(S) S_j
\]
\[
\leq \sum_{1 \leq j \leq d} S_j^{2k} \Psi_{m-1}(S) S_j^2
\]
\[
\leq \cdots
\]
\[
\leq \sum_{1 \leq j \leq d} S_j^{k_1} \Psi_{m-1}(S) S_j^{k_1}
\]
for every \( k = (k_1, \cdots, k_d) \in \mathbb{Z}^d_+ \). By the assumption in the statement (1) it follows that \( \Psi_{m-1}(S) \leq 0 \). Repeating the process as above we can prove that
\[
\Psi_l(S) \leq 0 \quad \forall \ l \in \{1, 2, \cdots, m\}.
\]
This yields the conclusion that the operator \( S \) is a joint \( m \)-hyperexpansive tuple. \( \square \)

A similar argument to the one above proves the following theorem:

**Theorem 2.3.** Let \( S = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d \) be a commuting tuple of operators. If \( S \) is a joint \( m \)-contractive tuple and satisfies the following conditions

1. \( S_j^{k_j} \to 0 \) strongly for all \( j \in \{1, \cdots, d\} \),
2. \( S_j^* \Psi_k(S) S_j \geq S_j^{2k} \Psi_k(S) S_j^2 \) for all \( j \in \{1, 2, \cdots, d\} \) and \( k \in \{1, \cdots, m-1\} \),

then \( T \) is \( m \)-hypercontractive tuple.

**Proposition 2.4.** Let \( S = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d \) be commuting tuple of operators such that \( S = (S_1/\|S_1\|, \cdots, S_d/\|S_d\|) \) is a joint 1-isometric tuple. The following properties hold.

1. If \( S \) is joint \( m \)-expansive tuple, then \( S \) is \( m \)-hyperexpansive tuple.
2. If \( S \) is joint \( m \)-contractive tuple, then \( S \) is a joint \( m \)-hypercontractive tuple.
Proof. By (2.9) we have for all \( k \in \{1, 2, \cdots, m\} \)
\[
Q_k(S, x) = Q_{k-1}(S, x) - \sum_{1 \leq j \leq d} Q_{k-1}(S_j, x), \quad \forall \ x \in \mathcal{H}.
\]

If \( S \) is a joint isometric tuple on \( \mathcal{R}(S) \), it is well known that \( S \) is an \( k \)-isometric tuple on \( \mathcal{R}(S) \) for \( k = 1, \cdots, m \). Consequently,
\[
Q_1(S, x) = Q_2(S, x) = \cdots = Q_{m-1}(S, x) = Q_m(S, x) = 0.
\]

If \( S \) is a joint \( m \)-expansive tuple, then it is a joint \((m-1)\)-expansive tuple. By repeating this process we get obtain that \( S \) is joint \( k \)-expansive tuple for \( k = 1, 2 \cdots, m \). Consequently, \( S \) is joint \( m \)-hyperexpansive.

By the same argument as above, (2) is obtained. \( \Box \)

A point \( \mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d \) is said to be a joint approximate point of \( S = (S_1, \cdots, S_d) \) if there exists a unit sequence of vectors \( (\xi_n)_n \subset \mathcal{H} \) such that
\[
\lim_{n \to \infty} \| (S_j - \mu_j)\xi_n \| = 0 \quad \text{for} \quad j = 1, \cdots, d.
\]
The joint approximate point spectrum of \( S \), denoted by \( \sigma_{ap}(S) \) is defined by
\[
\sigma_{ap}(S) := \{ \mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d / \mu \text{ is a joint approximate point spectrum of } S \}.
\]

We denote by
\[
\mathcal{B}(\mathbb{C}^d) := \{ \mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d / \| \mu \|_2 = \left( \sum_{1 \leq j \leq d} |\mu_j|^2 \right)^{\frac{1}{2}} < 1 \}
\]
and
\[
\partial \mathcal{B}(\mathbb{C}^d) := \{ \mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d / \| \mu \|_2 = \left( \sum_{1 \leq j \leq d} |\mu_j|^2 \right)^{\frac{1}{2}} = 1 \}.
\]

**Proposition 2.5.** Let \( S = (S_1, \cdots, S_d) \in \mathcal{B}(\mathcal{H})^d \) be an \( m \)-expansive tuple for some positive integer \( m \). The following properties hold.

1. If \( m \) is even, then the approximate point spectrum of \( S \) lies in the boundary of the unit ball of \( (\mathbb{C}^d, \| \cdot \|_2) \).
2. If \( m \) is odd, then \( \sigma_{ap}(S) \subset \mathbb{C}^d \setminus \mathcal{B}(\mathbb{C}^d) \) and \( \mathcal{N}(S) := \bigcap_{1 \leq j \leq d} \mathcal{N}(S_j) = \{0\} \).

**Proof.** Let \( \mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d \) is in the approximate point spectrum of \( S \), then there exists a sequence \( (\xi_n)_n \subset \mathcal{X} \) such that for all \( n \), \( \| \xi_n \| = 1 \) and \( \|(S_j - \mu_j)\xi_n\| \to 0 \) as \( n \to +\infty \). Thus for each integer \( \beta_j \in \mathbb{N} \), \( \lim_{n \to +\infty} \|(S_j^\beta - \mu_j^\beta)\xi_n\| = 0 \) and so that
\[
\lim_{n \to +\infty} \|(S_j^\beta - \mu_j^\beta)\xi_n\| = 0 \quad \forall \ \beta = (\beta_1, \cdots, \beta_d) \in \mathbb{Z}_+^d.
\]

Now it is easy to see that
\[
\|S^\beta \xi_n\|^2 \to |\mu|^{2\beta} \quad \text{as} \quad n \to \infty.
\]
Since $S$ is an $(m,p)$-expansive, it follows that
\[ 0 \geq \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \left( \sum_{|\alpha| = k} \frac{k!}{\alpha!} \| (S)^{\alpha} \xi_n \|^2 \right) \]
By taking $n \to \infty$ in the last inequality, we obtain
\[ 0 \geq \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} |\mu|^{2k} \]
\[ \geq \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\beta| = k} \frac{k!}{\beta!} |\mu|^{2k} \]
\[ \geq \left( 1 - \|\mu\|_2^2 \right)^m. \]
So,
\[ (1 - \|\mu\|_p^p)^m \leq 0 \]  \hspace{1cm} (2.10)
Now, we have that
(1) If $m$ is even, then $0 \leq (1 - \|\mu\|_p^p)^m \leq 0$ from (2.10), and so $\|\mu\|_2 = 1$. Thus means that $\sigma_{ap}(S) \subset \partial B(C^d)$ and so
\[ \partial \sigma(S) \subset \sigma_{ap}(S) \subset \partial B(C^d). \]
(2) Assume that $m$ is odd. If $\|\mu\|_2 < 1$, then $0 < (1 - \|\mu\|_2^2)^m \leq 0$ from (2.10), which is a contradiction. Hence,
\[ \sigma_{ap}(S) \subset C^d \setminus \mathbb{B}(C^d). \]
In particular, $(0, \cdots, 0) \notin \sigma_{ap}(S)$. Thus $\mathcal{N}(S) = \{0\}. \quad \Box$

REFERENCES

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