The notions of doubt $\mathcal{N}$-subalgebras and doubt $\mathcal{N}$-ideals in $BCK$-algebras are introduced, and related properties are investigated. Characterizations of a doubt $\mathcal{N}$-subalgebra and a doubt $\mathcal{N}$-ideal are given, and relations between them are discussed.

1. INTRODUCTION

The study of $BCK$-algebras was introduced by Imai and Iséki [10] in 1966. $BCK$-algebras have been applied to many branches of mathematics, such as functional analysis, group theory, topology, probability theory.

A crisp set $C$ in a universe $\mathcal{X}$ is a function $\lambda_C : \mathcal{X} \to \{0, 1\}$ yielding the value 0 for elements excluded from the set $C$ and the value 1 for elements belonging to the set $C$. As a generalization of crisp sets, Zadeh [17] introduced the degree of positive membership in 1965 and defined the fuzzy sets. Jun et al. [12] presented a new function which is called negative-valued function, and developed $\mathcal{N}$-structures as a one of the hybrid models of fuzzy sets. They applied $\mathcal{N}$-structures in $BCK$-algebras and proposed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals [12]. In [11], Jun established the definition of doubt fuzzy subalgebras and ideals in $BCK$-algebras. After that, many Hybrid models of fuzzy sets were applied in $BCK$-algebras and other algebraic structures [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

In this paper, we discuss an $\mathcal{N}$-structure with an application to $BCK$-algebras. We introduce the notions of doubt $\mathcal{N}$-subalgebras and doubt $\mathcal{N}$-ideals in $BCK$-algebras, and investigate related properties. Then, we present some characterizations of them by means of doubt level subset. Moreover, relations between a doubt $\mathcal{N}$-subalgebra and a doubt $\mathcal{N}$-ideal in $BCK$-algebras are discussed.

2. PRELIMINARIES

In this section, we include some basic definitions and preliminary facts about $BCK$-algebras which are essential for our results.

By a $BCK$-algebra, we mean an algebra $(\mathcal{X}, \ast, 0)$ of type $(2, 0)$ satisfying the following axioms for all $x, y, z \in \mathcal{X}$:

\[(1) \ (x \ast y) \ast (x \ast z) \ast (z \ast y) = 0,\]

2010 Mathematics Subject Classification. 06F35, 03G25, 03B52, 03B05.
Key words and phrases. $BCK$/$BCI$-algebra; $\mathcal{N}$-structures; Doubt $\mathcal{N}$-subalgebra; Doubt $\mathcal{N}$-ideal.
Received: February 29, 2020. Accepted: April 25, 2020.
*Corresponding author.
Definition 2.1. A fuzzy set

Definition 3.1. An investigate some of their properties.

Definition 2.3. An algebras as follows:

X

Definition 2.2. A fuzzy set

X

subalgebra of

x,y

∈X

is an N (briefly, N

We say that, an element of F

X

∗

x

∗

y

= 0

(II) (x * (x * y)) * y = 0,

(III) x * x = 0,

(IV) 0 * x = 0,

(V) x * y = 0 and y * x = 0 imply x = y.

Any BCK-algebra X satisfies the following axioms for all x, y, z \in X:

(I1) x * 0 = x,

(I2) (x * y) * z = (x * z) * y,

(I3) x * y \leq x,

(I4) (x * y) * z \leq (x * z) * (y * z),

(I5) x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x.

A partial ordering \leq on a BCK-algebra X can be defined by x \leq y if and only if

x * y = 0. A non-empty subset K of a BCK-algebra X is called a subalgebra of X if

x * y \in K, \forall x, y \in X, and an ideal of X if \forall x, y \in X,

(1) 0 \in K,

(2) x * y \in K and y \in K imply x \in K.

Definition 2.1. \[11\] A fuzzy set A = \{(x, \mu_A(x)) \mid x \in X\} in X is called a doubt fuzzy subalgebra of X if \mu_A(x * y) \leq \max\{\mu_A(x), \mu_A(y)\} for all x, y \in X.

Definition 2.2. \[11\] A fuzzy set A = \{(x, \mu_A(x)) \mid x \in X\} in X is called a doubt fuzzy ideal of X if \mu_A(0) \leq \mu_A(x) and \mu_A(x) \leq \max\{\mu_A(x * y), \mu_A(y)\} for all x, y \in X.

Denote by \mathcal{F}(X, [-1, 0]) the collection of functions from a set X to the interval [-1, 0]. We say that, an element of \mathcal{F}(X, [-1, 0]) is a negative-valued function from X to [-1, 0] (briefly, \mathcal{N}-function on X). By an \mathcal{N}-structure we mean an ordered pair (X, \varphi), where \varphi is an \mathcal{N}-function on X. In what follows, let X be a BCK-algebra and \varphi an \mathcal{N}-function on X unless otherwise specified.

In \[12\], Jun et al. introduced the concepts of \mathcal{N}-subalgebras and \mathcal{N}-ideals in BCK-algebras as follows:

Definition 2.3. An \mathcal{N}-structure (X, \varphi) is called an \mathcal{N}-subalgebra of X if for all x, y \in X :

\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\}.

Definition 2.4. An \mathcal{N}-structure (X, \varphi) is called an \mathcal{N}-ideal of X if for all x, y \in X :

(1) \varphi(0) \leq \varphi(x),

(2) \varphi(x) \leq \max\{\varphi(x * y), \varphi(y)\}.

3. Doubt \mathcal{N}-Subalgebras and \mathcal{N}-Ideals

In this section, we introduce doubt \mathcal{N}-subalgebras and ideals in BCK-algebras and investigate some of their properties.

Definition 3.1. An \mathcal{N}-structure (X, \varphi) is called a doubt \mathcal{N}-subalgebra of X if for all x, y \in X :

\varphi(x * y) \geq \min\{\varphi(x), \varphi(y)\}.
Example 3.2. Consider a $BCK$–algebra $X = \{0, a, b, c\}$ with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $(X, \varphi)$ be an $N$-structure in which $\varphi$ is given by

$$\varphi(x) = \begin{cases} -0.2, & \text{if } x = 0 \\ -0.3, & \text{if } x = a, b \\ -0.8, & \text{if } x = c. \end{cases}$$

By routine calculation, we know that $(X, \varphi)$ is a doubt $N$-subalgebra of $X$.

For any $N$-function $\varphi$ and $\alpha \in [-1, 0]$, we define the set

$$\varphi_\alpha = \{x \in X : \varphi(x) \geq \alpha\}.$$

Theorem 3.1. Let $(X, \varphi)$ be an $N$-structure over $X$ and let $\alpha \in [-1, 0]$. If $(X, \varphi)$ is a doubt $N$-subalgebra of $X$, then the nonempty set $\varphi_\alpha$ is a subalgebra of $X$.

Proof. Let $\alpha \in [-1, 0]$ and let $\varphi_\alpha \neq \emptyset$. If $x, y \in \varphi_\alpha$, then $\varphi(x) \geq \alpha$ and $\varphi(y) \geq \alpha$. It follows from Definition 3.1 that

$$\varphi(x \ast y) \geq \min\{\varphi(x), \varphi(y)\} \geq \alpha.$$

Hence, $x \ast y \in \varphi_\alpha$, and therefore $\varphi_\alpha$ is a subalgebra of $X$. \hfill \Box

Theorem 3.2. Let $(X, \varphi)$ be an $N$-structure over $X$ and assume that $\emptyset \neq \varphi_\alpha$ is a subalgebra of $X$ for all $\alpha \in [-1, 0]$. Then, $(X, \varphi)$ is a doubt $N$-subalgebra of $X$.

Proof. Assume that $\emptyset \neq \varphi_\alpha$ is a subalgebra of $X$ for all $\alpha \in [-1, 0]$. If there exist $a', b' \in X$ such that

$$\varphi(a' \ast b') < \min\{\varphi(a'), \varphi(b')\}.$$

Then by taking

$$\alpha_\alpha = \frac{1}{2}[\varphi(a' \ast b') + \min\{\varphi(a'), \varphi(b')\}],$$

we have

$$\varphi(a' \ast b') < \alpha_\alpha < \min\{\varphi(a'), \varphi(b')\}.$$

Hence, $a' \ast b' \notin \varphi_{\alpha_\alpha}, a' \in \varphi_{\alpha_\alpha}$ and $b' \in \varphi_{\alpha_\alpha}$. This is a contradiction. Therefore, $(X, \varphi)$ is a doubt $N$-subalgebra of $X$. \hfill \Box

Theorem 3.3. If $(X, \varphi)$ is a doubt $N$-subalgebra of $X$, then $\varphi(0) \geq \varphi(x)$ for all $x \in X$.

Proof. For any $x \in X$, we have $\varphi(0) = \varphi(x \ast x) \geq \min\{\varphi(x), \varphi(x)\} = \varphi(x)$. This completes the proof. \hfill \Box

Theorem 3.4. If every doubt $N$-subalgebra $(X, \varphi)$ of $X$, satisfies

$$\varphi(x \ast y) \geq \varphi((y))$$

for all $x, y \in X$, then $(X, \varphi)$ is constant.

Proof. Note that in a $BCK$-algebra $X, x \ast 0 = x$ for all $x \in X$. Since $\varphi(x \ast y) \geq \varphi((y))$, we have
\[ \varphi(x) = \varphi(x \ast 0) \geq \varphi(0). \]

It follows from Theorem 3.3 that \( \varphi(x) = \varphi(0) \) for all \( x, y \in X \). Therefore, \( (X, \varphi) \) is constant. \( \square \)

Now, we introduce the notion of doubt \( N \)-ideals in \( BCK \)-algebras.

**Definition 3.3.** An \( N \)-structure \( (X, \varphi) \) is called a doubt \( N \)-ideal of \( X \) if for all \( x, y \in X \):

1. \( \varphi(0) \geq \varphi(x), \)
2. \( \varphi(x) \geq \min\{\varphi(x \ast y), \varphi(y)\}. \)

**Example 3.4.** Consider a \( BCK \)-algebra \( X = \{0, a, b, c\} \) which is given in Example 3.6. Let \( (X, \varphi) \) be an \( N \)-structure in which \( \varphi \) is defined by

\[ \varphi(x) = \begin{cases} -0.2, & \text{if } x = 0 \\ -0.3, & \text{if } x = a, b \\ -0.8, & \text{if } x = c. \end{cases} \]

By routine calculation, we know that \( (X, \varphi) \) is a doubt \( N \)-ideal of \( X \).

**Theorem 3.5.** Let \( (X, \varphi) \) be a doubt \( N \)-ideal of \( X \). If \( \leq \) is a partial ordering on \( X \), then \( \varphi(x) \geq \varphi(y) \) for all \( x, y \in X \) such that \( x \leq y \).

**Proof.** Let \( (X, \varphi) \) be a doubt \( N \)-ideal of \( X \). It is known that \( \leq \) is a partial ordering on \( X \) defined by \( x \leq y \) if and only if \( x \ast y = 0 \) for all \( x, y \in X \). Then,

\[
\varphi(x) \geq \min\{\varphi(x \ast y), \varphi(y)\} = \min\{\varphi(0), \varphi(y)\} = \varphi(y).
\]

This completes the proof. \( \square \)

**Theorem 3.6.** Let \( (X, \varphi) \) be a doubt \( N \)-ideal of \( X \). Then,

\[
\varphi(x \ast y) \geq \varphi((x \ast y) \ast y) \iff \varphi((x \ast z) \ast (y \ast z)) \geq \varphi((x \ast y) \ast z)
\]

for all \( x, y, z \in X \).

**Proof.** Note that

\[
((x \ast (y \ast z)) \ast z) \ast z = ((x \ast z) \ast (y \ast z)) \ast z \\
\leq (x \ast y) \ast z
\]

for all \( x, y, z \in X \). Assume that \( \varphi(x \ast y) \geq \varphi((x \ast y) \ast y) \) for all \( x, y, z \in X \). It follows from (12) and Theorem 3.5 that

\[
\varphi((x \ast z) \ast (y \ast z)) = \varphi((x \ast (y \ast z)) \ast z) \\
\geq \varphi(((x \ast (y \ast z)) \ast z) \ast z) \\
\geq \varphi((x \ast y) \ast z),
\]

for all \( x, y, z \in X \).

Conversely, suppose that

\[
\varphi((x \ast z) \ast (y \ast z)) \geq \varphi((x \ast y) \ast z)
\]

for all \( x, y, z \in X \). If we substitute \( z \) for \( y \) in Equation (3.1), then

\[
\varphi(x \ast z) = \varphi((x \ast z) \ast 0) = \varphi((x \ast z) \ast (z \ast z)) \geq \varphi((x \ast z) \ast z),
\]
Theorem 3.7. Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. Then,
\[ \varphi(x * y) \geq \min\{\varphi(x), \varphi(z + y)\} \]
for all $x, y, z \in \mathcal{X}$.

Proof. Note that $((x + y) * (x + z)) \leq (z + y)$ for all $x, y, z \in \mathcal{X}$. It follows from Theorem 3.5 that
\[ \varphi((x + y) * (x + z)) \geq \varphi(z + y). \]

Now, by Definition 3.3, we have
\[ \varphi(x * y) \geq \min\{\varphi((x + y) * (x + z)), \varphi(x + z)\} \]
\[ \geq \min\{\varphi(x + z), \varphi(z + y)\} \]
for all $x, y, z \in \mathcal{X}$. This completes the proof.

Theorem 3.8. Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. Then,
\[ \varphi(x * (x + y)) \geq \varphi(y) \]
for all $x, y \in \mathcal{X}$.

Proof. Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. Then, for all $x, y \in \mathcal{X}$, we have
\[ \varphi(x * (x + y)) \geq \min\{\varphi((x * (x + y)) * y), \varphi(y)\} \]
\[ = \min\{\varphi((x * y) * (x + y)), \varphi(y)\} \]
\[ = \min\{\varphi(0), \varphi(y)\} \]
\[ = \varphi(y). \]

This completes the proof.

Theorem 3.9. Let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure over $\mathcal{X}$ and let $\alpha \in [-1, 0]$. If $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$, then the nonempty set $\varphi_{\alpha}$ is an ideal of $\mathcal{X}$.

Proof. Assume that $\varphi_{\alpha} \neq \emptyset$ for $\alpha \in [-1, 0]$. Clearly, $0 \in \varphi_{\alpha}$. Let $x * y \in \varphi_{\alpha}$ and $y \in \varphi_{\alpha}$. Then, $\varphi(x + y) \geq \alpha$ and $\varphi(y) \geq \alpha$. It follows from Definition 3.3 that
\[ \varphi(x) \geq \min\{\varphi(x + y), \varphi(y)\} \geq \alpha, \]
so, $x \in \varphi_{\alpha}$. Therefore, $\varphi_{\alpha}$ is an ideal of $\mathcal{X}$.

Theorem 3.10. Let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure over $\mathcal{X}$ and assume that $\emptyset \neq \varphi_{\alpha}$ is an ideal of $\mathcal{X}$ for all $\alpha \in [-1, 0]$. Then, $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$.

Proof. Assume that $\emptyset \neq \varphi_{\alpha}$ is an ideal of $\mathcal{X}$ for all $\alpha \in [-1, 0]$. For any $x \in \mathcal{X}$, let $\varphi(x) = \alpha$. Then, $x \in \varphi_{\alpha}$, and so $\varphi_{\alpha}$ is nonempty. Since $\varphi_{\alpha}$ is an ideal of $\mathcal{X}$, so $0 \in \varphi_{\alpha}$. Hence, $\varphi(0) \geq \alpha = \varphi(x)$ for all $x \in \mathcal{X}$. If there exists $a', b' \in \mathcal{X}$ such that
\[ \varphi(a') < \min\{\varphi(a' * b'), \varphi(b')\} \]
Then, by taking
\[ \alpha_1 = \frac{1}{2}[\varphi(a') + \min\{\varphi(a' * b'), \varphi(b')\}], \]
we have
\[ \varphi(a') < \alpha_1 < \min\{\varphi(a' * b'), \varphi(b')\}. \]
Hence, \( a' \neq \varphi_{\alpha_1}, \alpha' * b' \in \varphi_{\alpha_1} \) and \( b' \in \varphi_{\alpha_1} \). This is a contradiction, and so \( \varphi(x) \geq \min\{\varphi(x * y), \varphi(y)\} \) for all \( x, y \in X \). Therefore, \((X, \varphi)\) is a doubt \( \mathcal{N} \)-ideal of \( X \). \( \square \)

**Theorem 3.11.** Let \((X, \varphi)\) be an \( \mathcal{N} \)-structure over \( X \). If the inequality \( x * y \leq z \) holds in \( X \), then \( \varphi(x) \geq \min\{\varphi(y), \varphi(z)\} \) for all \( x, y, z \in X \).

**Proof.** \((X, \varphi)\) be an \( \mathcal{N} \)-structure over \( X \) and \( x, y, z \in X \) be such that \( x * y \leq z \). Then, \((x * y) * z = 0 \), and so

\[
\varphi(x) \geq \min\{\varphi(x * y), \varphi(y)\} \\
geq \min\{\min\{\varphi((x * y) * z), \varphi(z)\}, \varphi(y)\} \\
= \min\{\min\{\varphi(0), \varphi(z)\}, \varphi(y)\} \\
= \min\{\varphi(y), \varphi(z)\}.
\]

This completes the proof. \( \square \)

**Theorem 3.12.** Every doubt \( \mathcal{N} \)-ideal of \( X \) is a doubt \( \mathcal{N} \)-subalgebra of \( X \).

**Proof.** Let \((X, \varphi)\) be a doubt \( \mathcal{N} \)-ideal of \( X \). For any \( x, y \in X \), we have

\[
\varphi(x * y) \geq \min\{\varphi((x * y) * x), \varphi(x)\} \\
= \min\{\varphi((x * x) * y), \varphi(x)\} \\
= \min\{\varphi(0 * y), \varphi(x)\} \\
= \min\{\varphi(0), \varphi(x)\} \\
\geq \min\{\varphi(x), \varphi(y)\}
\]

Hence, \((X, \varphi)\) is a doubt \( \mathcal{N} \)-subalgebra of \( X \). \( \square \)

**Example 3.5.** In Example 3.4, \((X, \varphi)\) is a doubt \( \mathcal{N} \)-ideal of \( X \), so that \((X, \varphi)\) is a doubt \( \mathcal{N} \)-subalgebra of \( X \).

The converse of Theorem 3.12 is not true in general.

**Example 3.6.** Consider a \( BCK \)-algebra \( X = \{0, a, b, c, d\} \) with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Let \((X, \varphi)\) be an \( \mathcal{N} \)-structure in which \( \varphi \) is given by

\[
\varphi(x) = \begin{cases} 
0.0, & \text{if } x = 0 \\
-0.2, & \text{if } x = a \\
-0.6, & \text{if } x = b \\
-0.4, & \text{if } x = c \\
-0.8, & \text{if } x = d.
\end{cases}
\]

Then, \((X, \varphi)\) is a doubt \( \mathcal{N} \)-subalgebra of \( X \), but it is not a doubt \( \mathcal{N} \)-ideal of \( X \) since \( \varphi(d) = -0.8 < -0.4 = \min\{\varphi(d * c), \varphi(c)\} \).

We give a condition for a doubt \( \mathcal{N} \)-subalgebra to be a doubt \( \mathcal{N} \)-ideal in a \( BCK \)-algebra.

**Theorem 3.13.** Let \((X, \varphi)\) be a doubt \( \mathcal{N} \)-subalgebra of \( X \). If the inequality \( x * y \leq z \) holds in \( X \), then \((X, \varphi)\) is a doubt \( \mathcal{N} \)-ideal of \( X \).
Proof. Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-subalgebra of $\mathcal{X}$. Then, from Theorem 3.3, $\varphi(0) \geq \varphi(x)$, for all $x \in \mathcal{X}$. As $x \ast y \leq z$ holds in $\mathcal{X}$, then from Theorem 3.11 we get $\varphi(x) \geq \min\{\varphi(y), \varphi(z)\}$ for all $x, y, z \in \mathcal{X}$.

Since $x \ast (x \ast y) \leq y$ for all $x, y \in \mathcal{X}$, then $\varphi(x) \geq \min\{\varphi(x \ast y), \varphi(y)\}$. Hence, $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$.

Theorem 3.14. Let $(\mathcal{X}, \varphi)$ be a doubt $\mathcal{N}$-ideal of $\mathcal{X}$. Then, the set

$$H = \{x \in \mathcal{X} : \varphi(x) = \varphi(0)\}$$

is an ideal of $\mathcal{X}$.

Proof. Obviously, $0 \in H$. Hence, $H \neq \emptyset$. Now, let $x, y \in H$ such that $x \ast y, y \in H$. Then, $\varphi(x \ast y) = \varphi(0) = \varphi(y)$. Now, $\varphi(x) \geq \min\{\varphi(x \ast y), \varphi(y)\} = \varphi(0)$. Since $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$, $\varphi(0) \geq \varphi(x)$. Therefore, $\varphi(0) = \varphi(x)$. It follows that $x \in H$, for all $x, y \in \mathcal{X}$. Therefore, $H$ is an ideal of $\mathcal{X}$. □

For any element $\omega_{\alpha} \in X$, we consider the set:

$$\varphi_{\omega_{\alpha}} = \{x \in \mathcal{X} : \varphi(x) \geq \varphi(\omega_{\alpha})\}.$$  

Clearly, $\omega_{\alpha} \in \varphi_{\omega_{\alpha}}$. So that $\omega_{\alpha} \in \varphi_{\omega_{\alpha}}$ is a nonempty set of $\mathcal{X}$.

Theorem 3.15. Let $\omega_{\alpha}$ be any element of $\mathcal{X}$. If $(\mathcal{X}, \varphi)$ is a doubt $\mathcal{N}$-ideal of $\mathcal{X}$, then $\varphi_{\omega_{\alpha}}$ is an ideal of $\mathcal{X}$.

Proof. Clearly, $0 \in \varphi_{\omega_{\alpha}}$. Let $x, y \in \mathcal{X}$ be such that $x \ast y \in \varphi_{\omega_{\alpha}}$ and $y \in \varphi_{\omega_{\alpha}}$. Then, $\varphi(x \ast y) \geq \varphi(\omega_{\alpha})$ and $\varphi(y) \geq \varphi(\omega_{\alpha})$. It follows that from Definition 3.3 that

$$\varphi(x) \geq \min\{\varphi(x \ast y), \varphi(y)\} \geq \varphi(\omega_{\alpha}).$$

Hence, $x \in \varphi_{\omega_{\alpha}}$, and therefore $\varphi_{\omega_{\alpha}}$ is an ideal of $\mathcal{X}$. □

Theorem 3.16. Let $\omega_{\alpha} \in \mathcal{X}$ and let $(\mathcal{X}, \varphi)$ be an $\mathcal{N}$-structure over $\mathcal{X}$. Then,

1. If $\varphi_{\omega_{\alpha}}$ is an ideal of $\mathcal{X}$, then the following assertion is valid for all $x, y, z \in \mathcal{X}:
   (A1) $\varphi(z) \leq \min\{\varphi(y \ast z), \varphi(z)\}$
   (A2) $\varphi(\omega_{\alpha}) \geq \varphi(x)$

2. If $(\mathcal{X}, \varphi)$ satisfies (A1) and
   (A2) $\varphi(0) \geq \varphi(\omega_{\alpha})$

for all $x \in \mathcal{X}$. Then, $\varphi_{\omega_{\alpha}}$ is an ideal for all $\omega_{\alpha} \in Im(\varphi)$.

Proof. (1) Assume that $\varphi_{\omega_{\alpha}}$ is an ideal of $\mathcal{X}$ for $\omega_{\alpha} \in \mathcal{X}$. Let $x, y, z \in \mathcal{X}$ be such that $\varphi(x) \leq \min\{\varphi(y \ast z), \varphi(z)\}$. Then, $y \ast z \in \varphi_{\omega_{\alpha}}$ and $z \in \varphi_{\omega_{\alpha}}$, where $\omega_{\alpha} = x$. Since $\varphi_{\omega_{\alpha}}$ is an ideal of $\mathcal{X}$, it follows that $y \in \varphi_{\omega_{\alpha}}$ for $\omega_{\alpha} = x$. Hence, $\varphi(y) \geq \varphi(\omega_{\alpha}) = \varphi(x)$.

(2) Let $\omega_{\alpha} \in Im(\varphi)$ and suppose that $(\mathcal{X}, \varphi)$ satisfies (A1) and (A2). Clearly, $0 \in \varphi_{\omega_{\alpha}}$ by (A2). Let $x, y \in \mathcal{X}$ be such that $x \ast y \in \varphi_{\omega_{\alpha}}$ and $y \in \varphi_{\omega_{\alpha}}$. Then, $\varphi(x \ast y) \geq \varphi(\omega_{\alpha})$ and $\varphi(y) \geq \varphi(\omega_{\alpha})$, so

$$\min\{\varphi(x \ast y), \varphi(y)\} \geq \varphi(\omega_{\alpha}).$$

It follows from (A1) that $\varphi(\omega_{\alpha}) \leq \varphi(x)$. Thus, $x \in \varphi_{\omega_{\alpha}}$, and therefore $\varphi_{\omega_{\alpha}}$ is an ideal of $\mathcal{X}$. □
4. Conclusions

Doubt $\mathcal{N}$-subalgebras and doubt $\mathcal{N}$-ideals with special properties play an important role in investigating the structure of an algebraic system. In this work, we discussed an $\mathcal{N}$-structure with an application to $BCK$-algebras. We introduced the notions of doubt $\mathcal{N}$-subalgebras and doubt $\mathcal{N}$-ideals in $BCK$-algebras, and investigated related properties. We considered some characterizations of a doubt $\mathcal{N}$-subalgebra and a doubt $\mathcal{N}$-ideal in $BCK$-algebras by means of doubt level subset. Relations between a doubt $\mathcal{N}$-subalgebra and a doubt $\mathcal{N}$-ideal were provided. We believe that our results presented in this paper will give a foundation for further study the algebraic structure of $BCK$-algebras.

5. Acknowledgements

The authors are thankful to the editors and the anonymous reviewers for their valuable suggestions and comments on the manuscript.

References

ABD GHAFUR AHMAD
School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor DE, Malaysia
Email address: ghafur@ukm.edu.my

G. MUHIUDDIN
Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia
Email address: chishtygm@gmail.com