A NEW FORM OF GENERALIZED m-PF IDEALS IN BCK/BCI-ALGEBRAS

ANAS AL-MASARWAH * AND ABD GHAFUR AHMAD

ABSTRACT. In this paper, we introduce a new kind of an m-polar fuzzy ideal of a BCK/BCI-algebra called, an m-polar \((\in, \in \lor q)\) fuzzy ideal and investigate some of its properties. Ordinary ideals and m-polar \((\in, \in \lor q)\) fuzzy ideals are connected by means of level cut subset.

1. INTRODUCTION

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [12], and they have been applied to several domains of mathematics, such as group theory, topology, functional analysis and probability theory. Additionally, Iséki [13] initiated the idea of a BCI-algebra, which is a generalization of a BCK-algebra.

The idea of fuzzy sets was introduced by Zadeh [20] in 1965 to handle uncertainties in several real applications, and the idea of bipolar fuzzy sets on a universe \(X\) was introduced by Zhang [21] in 1994 as a generalization of fuzzy sets. The notion of m-polar fuzzy sets was presented by Chen et al. [9] in 2014 as an extension of bipolar fuzzy sets. Bipolar fuzzy sets, m-polar fuzzy (m-PF) sets and several hybrid models of fuzzy sets play a prominent rule in several algebraic structures, such as BCK/BCI-algebras [3, 7, 4, 6, 18, 16, 17], hemirings [11], groups [10] and lie-subalgebras [12]. In 1971, Rosenfeld [19] applied fuzzy sets to groups and proposed the concept of fuzzy subgroups. As a generalization of fuzzy subgroups, Bhakat and Das [8] initiated the notions of \((\in, \in \lor q)\)-fuzzy subgroups by using the concept of fuzzy points and its “belongingness (\(\in\))” and “quasi-coincidence (\(q\))” with a fuzzy set. Jun [15] introduced a generalization of fuzzy ideals in BCK/BCI-algebras, called \((\alpha, \beta)\)-fuzzy ideals. After that, Jana et al. [14] proposed the concept of \((\in, \in \lor q)\)-bipolar fuzzy ideals in BCK/BCI-algebras. Further, Al-Masarwah and Ahmad [5] presented the notion of m-polar \((\in, \in \lor q)\)-fuzzy ideals in BCK/BCI-algebras as a generalization of m-polar fuzzy ideals.

This paper is a continuation of papers [4] and [5]. We introduce the notion of m-polar \((\in, \in \lor q)\) fuzzy ideals and investigate some of its properties. We discuss the relation between ordinary ideals and m-polar \((\in, \in \lor q)\) fuzzy ideals in BCK/BCI-algebras.

2010 Mathematics Subject Classification. 06F35, 03G25, 03B52, 03B05.
Key words and phrases. BCK/BCI-algebra; m-polar fuzzy ideal; m-polar \((\in, \in \lor q)\) fuzzy ideal.
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2. Preliminaries

Some of the significant notions pertaining to $BCK/BCI$-algebras, $m$-PF sets, $m$-PF points and $m$-PF ideals that are useful for subsequent discussions are stated below. In what follows, let $X$ be a $BCK/BCI$-algebra unless otherwise specified.

By a $BCI$-algebra, we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following axioms for all $x, y, z \in X$:

1. $(x * y) * (x * z) * (z * y) = 0$,
2. $(x * (x * y)) * y = 0$,
3. $x * x = 0$,
4. $x * y = 0$ and $y * x = 0$ imply $x = y$.

If a $BCI$-algebra $X$ satisfies $0 * x = 0 \forall x \in X$, then $X$ is called a $BCK$-algebra. A partial ordering $\leq$ on a $BCK/BCI$-algebra $X$ can be defined by $x \leq y$ if and only if $x * y = 0$. Any $BCK/BCI$-algebra $X$ satisfies the following axioms for all $x, y, z \in X$:

1. $x * 0 = x$,
2. $(x * y) * z = (x * z) * y$.

A non-empty subset $J$ of $X$ is said to be an ideal of $X$ if for all $x, y \in X$:

1. $0 \in J$,
2. $x * y \in J$ and $y \in J$ imply $x \in J$.

**Definition 2.1** ([9]). An $m$-PF set $\widehat{H}$ on $X(\neq \emptyset)$ is a function $\widehat{H} : X \to [0, 1]^m$, where

$$\widehat{H}(x) = (p_1 \circ \widehat{H}(x), p_2 \circ \widehat{H}(x), ..., p_m \circ \widehat{H}(x))$$

is the membership value of every element $x \in X$ and $p_i \circ \widehat{H} : [0, 1]^m \to [0, 1]$ is the $i$-th projection mapping for all $i = 1, 2, ..., m$. The values $\widehat{0} = (0, 0, ..., 0)$ and $\widehat{1} = (1, 1, ..., 1)$ are the smallest and largest values in $[0, 1]^m$, respectively.

**Definition 2.2** ([3]). An $m$-PF set $\widehat{H}$ of $X$ is said to be an $m$-PF ideal if the assertions below are valid: for all $x, y \in X$,

1. $\widehat{H}(0) \geq \widehat{H}(x)$,
2. $\widehat{H}(x) \geq \inf\{\widehat{H}(x * y), \widehat{H}(y)\}$.

That is,

1. $p_i \circ \widehat{H}(0) \geq p_i \circ \widehat{H}(x)$,
2. $p_i \circ \widehat{H}(x) \geq \inf\{p_i \circ \widehat{H}(x * y), p_i \circ \widehat{H}(y)\}$

for all $i = 1, 2, ..., m$.

**Definition 2.3** ([3]). Let $\widehat{H}$ be an $m$-PF set of $X$. Then, the set

$$\widehat{H}_t = \{x \in X \mid \widehat{H}(x) \geq \widehat{t}\}$$

is called the level cut subset of $\widehat{H}$ for all $\widehat{t} \in (0, 1]^m$.

An $m$-PF set $\widehat{H}$ of $X$ of the form

$$\widehat{H}(y) = \begin{cases} \widehat{t} = (t_1, t_2, ..., t_m) \in (0, 1]^m, & \text{if } y = x \\ \widehat{0} = (0, 0, ..., 0), & \text{if } y \neq x \end{cases}$$

is called an $m$-PF point, denoted by $x_{\widehat{t}}$, with support $x$ and value $(t_1, t_2, ..., t_m) = \widehat{t}$.

An $m$-PF point $x_{\widehat{t}}$

1. Belongs to $\widehat{H}$, denoted by $x_{\widehat{t}} \in \widehat{H}$, if $\widehat{H}(x) \geq \widehat{t}$ i.e., $p_i \circ \widehat{H}(x) \geq t_i$ for each $i = 1, 2, ..., m$. 

(2) Is quasi-coincident with \( \hat{H} \), denoted by \( x \triangleleft q \hat{H} \), if \( \hat{H}(x) + \hat{t} > \hat{1} \) i.e., \( p_{i} \circ \hat{H}(x) + t_{i} > \hat{1} \) for each \( i = 1, 2, \ldots, m \).

We say that

1. \( x \triangleright \alpha \hat{H} \) if \( x \triangleright \alpha \hat{H} \) does not hold,
2. \( x \in \triangledown q \hat{H} \) if \( x \in \hat{H} \) or \( x \triangleleft q \hat{H} \),
3. \( x \in \wedge q \hat{H} \) if \( x \in \hat{H} \) and \( x \triangleleft q \hat{H} \).

**Definition 2.4**. An \( m \)-PF set \( \hat{H} \) of \( X \) is called an \( m \)-polar \((\varepsilon, \in \triangledown q)\)-fuzzy ideal of \( X \) if the assertions below are valid: for all \( x, y \in X \) and \( \hat{t}, \hat{s} \in (0, 1)^{m} \),

1. \( x \triangleright \hat{t} \in \hat{H} \) implies \( 0 \triangleright \hat{t} \in \triangledown q \hat{H} \),
2. \( (x \ast y) \triangleright \hat{t} \in \hat{H} \) and \( y \triangleright \hat{s} \in \hat{H} \) imply \( x_{\inf(\hat{t}, \hat{s})} \in \triangledown q \hat{H} \).

3. \( m \)-POLAR \((\varepsilon, \varepsilon \triangledown q)\)-FUZZY IDEALS

In this section, we define \( m \)-polar \((\varepsilon, \varepsilon \triangledown q)\)-fuzzy ideals of \( X \) and discuss several results.

**Definition 3.1**. An \( m \)-PF set \( \hat{H} \) of \( X \) is called an \( m \)-polar \((\varepsilon, \varepsilon \triangledown q)\)-fuzzy ideal of \( X \) if the assertions below are valid: for all \( x, y \in X \) and \( \hat{t}, \hat{s} \in (0, 1)^{m} \),

1. \( 0 \triangleright \varepsilon \hat{H} \) implies \( x \triangleright \varepsilon \hat{H} \),
2. \( x_{\inf(\hat{t}, \hat{s})} \triangleright \varepsilon \hat{H} \) implies \( (x \ast y) \triangleright \varepsilon \hat{H} \) or \( y \triangleright \varepsilon \hat{H} \).

**Example 3.2**. Consider a \( BCK \)-algebra \( X = \{0, a, b, c, d\} \) which is defined in Table 1.

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Let \( \hat{H} \) be a 3-PF set defined as:

\[
\hat{H}(x) = \begin{cases} 
(0.91, 0.91, 0.97), & \text{if } x = 0 \\
(0.37, 0.37, 0.78), & \text{if } x = a \\
(0.50, 0.50, 0.50), & \text{if } x = b \\
(0.44, 0.44, 0.49), & \text{if } x = c \\
(0.25, 0.25, 0.28), & \text{if } x = d.
\end{cases}
\]

Clearly, \( \hat{H} \) is a 3-polar \((\varepsilon, \varepsilon \triangledown q)\)-fuzzy ideal of \( X \).

**Theorem 3.1**. An \( m \)-PF set \( \hat{H} \) of \( X \) is an \( m \)-polar \((\varepsilon, \varepsilon \triangledown q)\)-fuzzy ideal of \( X \) if and only if for all \( x, y \in X \):

(i) \( \sup\{\hat{H}(0), 0.5\} \geq \hat{H}(x) \),
(ii) \( \sup\{\hat{H}(x), 0.5\} \geq \inf\{\hat{H}(x \ast y), \hat{H}(y)\} \).

**Proof**. Let \( \hat{H} \) be an \( m \)-polar \((\varepsilon, \varepsilon \triangledown q)\)-fuzzy ideal of \( X \). Suppose there exists \( x \in X \) such that \( \sup\{\hat{H}(0), 0.5\} < \hat{t} = \hat{H}(x) \). Then,
\hat{t} \in (0.5, 1]^{m}, 0_{1, \mathcal{H}}, \text{ and } x_{\hat{t}} \in \hat{\mathcal{H}}.

By Definition 3.1 (1), we have \( x_{\hat{t}} \hat{\in} \mathcal{H}, \) i.e., \( \hat{\mathcal{H}}(x) < \hat{t} \) or \( \hat{\mathcal{H}}(x) + \hat{t} \leq \hat{1} \). Since \( \hat{\mathcal{H}}(x) = \hat{t} \), therefore \( \hat{t} \leq 0.5 \). This is a contradiction. Hence, \( \sup \{ \hat{\mathcal{H}}(0), 0.5 \} \geq \hat{\mathcal{H}}(x) \) for all \( x \in X \).

Suppose there exist \( x, y \in X \) such that \( \sup \{ \hat{\mathcal{H}}(x), 0.5 \} < \hat{t} = \inf \{ \hat{\mathcal{H}}(x+y), \hat{\mathcal{H}}(y) \} \). Then,

\[ \hat{t} \in (0.5, 1]^{m}, x_{\hat{t}} \in \hat{\mathcal{H}} \text{ and } (x \ast y)_{\hat{\mathcal{H}}} \in \hat{\mathcal{H}}. \]

It follows that \( (x \ast y)_{\mathcal{H}} \) or \( y_{\mathcal{H}} \). Then, \( \hat{\mathcal{H}}(x+y) + \hat{t} \leq \hat{1} \) or \( \hat{\mathcal{H}}(y) + \hat{t} \leq \hat{1} \). Since \( \hat{\mathcal{H}}(x+y) \geq \hat{t} \) and \( \hat{\mathcal{H}}(y) \geq \hat{t} \), it follows that \( \hat{t} \leq 0.5 \), a contradiction. Hence, \( \sup \{ \hat{\mathcal{H}}(x), 0.5 \} \geq \inf \{ \hat{\mathcal{H}}(x \ast y), \hat{\mathcal{H}}(y) \} \) for all \( x, y \in X \).

Conversely, let \( 0_{1} \in \mathcal{H} \). Then, \( \hat{\mathcal{H}}(0) < \hat{t} \), either \( \hat{\mathcal{H}}(0) \geq \hat{\mathcal{H}}(x) \) or \( \hat{\mathcal{H}}(0) < \hat{\mathcal{H}}(x) \). If \( \hat{\mathcal{H}}(0) \geq \hat{\mathcal{H}}(x) \), then \( \hat{\mathcal{H}}(x) < \hat{t} \), and so \( x_{\hat{t}} \in \hat{\mathcal{H}}. \) That is, \( x_{\hat{t}} \in \mathcal{H} \). If \( \hat{\mathcal{H}}(0) < \hat{\mathcal{H}}(x) \), then by (i), \( \hat{\mathcal{H}}(x) \leq 0.5 \). We consider two cases:

**Case (1).** If \( \hat{\mathcal{H}}(x) < \hat{t} \), then \( x_{\hat{t}} \in \mathcal{H} \), and so \( x_{\hat{t}} \in \mathcal{H} \) or \( y_{\mathcal{H}} \).

**Case (2).** If \( \hat{\mathcal{H}}(x) \geq \hat{t} \), then \( \hat{t} \leq \hat{\mathcal{H}}(x) \leq 0.5 \), it follows that \( x_{\hat{t}} \in \mathcal{H} \), and so \( x_{\hat{t}} \in \mathcal{H} \) or \( y_{\mathcal{H}} \).

Again, let \( x_{\inf \{ \hat{t}, \hat{s} \}} \in \mathcal{H} \) for \( \hat{t}, \hat{s} \in [0, 1] \). Then, \( \hat{\mathcal{H}}(x) < \inf \{ \hat{t}, \hat{s} \} \). We consider two cases:

**Case (1).** If \( \hat{\mathcal{H}}(x) \geq 0.5 \), then

\[ \inf \{ \hat{\mathcal{H}}(x \ast y), \hat{\mathcal{H}}(y) \} \leq \hat{\mathcal{H}}(x) \leq \inf \{ \hat{t}, \hat{s} \}. \]

Consequently, \( \hat{\mathcal{H}}(x \ast y) < \hat{t} \) or \( \hat{\mathcal{H}}(y) < \hat{s} \). That is, \( (x \ast y)_{\hat{\mathcal{H}}} \in \mathcal{H} \) or \( y_{\mathcal{H}} \). Hence, \( (x \ast y)_{\hat{\mathcal{H}}} \in \mathcal{H} \) or \( y_{\mathcal{H}} \).

**Case (2).** If \( \hat{\mathcal{H}}(x) < 0.5 \), then

\[ \inf \{ \hat{\mathcal{H}}(x \ast y), \hat{\mathcal{H}}(y) \} \leq 0.5. \]

Assume \( (x \ast y)_{\hat{t}} \in \mathcal{H} \) or \( y_{\hat{t}} \in \mathcal{H} \). Then, \( \hat{t} \leq \hat{\mathcal{H}}(x \ast y) \leq 0.5 \) or \( \hat{t} \leq \hat{\mathcal{H}}(y) \leq 0.5 \). Thus, \( \hat{\mathcal{H}}(x \ast y) + \hat{t} \leq 0.5 + 0.5 = \hat{1} \) or \( \hat{\mathcal{H}}(y) + \hat{t} \leq 0.5 + 0.5 = \hat{1} \). It follows that \( (x \ast y)_{\mathcal{H}} \) or \( y_{\mathcal{H}} \) and so \( (x \ast y)_{\hat{\mathcal{H}}} \in \mathcal{H} \) or \( y_{\mathcal{H}} \). Hence, \( \mathcal{H} \) is an \( m \)-polar \((\mathcal{E}, \mathcal{E} \vee \mathcal{Q})\)-fuzzy ideal of \( X \).

**Theorem 3.2.** Any \( m \)-polar \((\mathcal{E}, \mathcal{E} \vee \mathcal{Q})\)-fuzzy ideal \( \hat{\mathcal{H}} \) of \( X \) satisfies: for all \( x, y \in X \),

1. \( x \leq y \Rightarrow \sup \{ \hat{\mathcal{H}}(x), 0.5 \} \geq \hat{\mathcal{H}}(y) \),
2. \( x \ast y \leq z \Rightarrow \sup \{ \hat{\mathcal{H}}(x), 0.5 \} \geq \inf \{ \hat{\mathcal{H}}(y), \hat{\mathcal{H}}(z) \} \).

**Proof.** (1) Suppose that \( x \leq y \) for all \( x, y \in X \). Then, \( x \ast y = 0 \). We have

\[ \sup \{ \hat{\mathcal{H}}(x), 0.5 \} \geq \inf \{ \hat{\mathcal{H}}(x \ast y), \hat{\mathcal{H}}(y) \} = \inf \{ \hat{\mathcal{H}}(0), \hat{\mathcal{H}}(y) \} \geq \hat{\mathcal{H}}(y). \]

(2) Assume that \( x \ast y \leq z \) hold in \( X \). Then,

\[ \sup \{ \hat{\mathcal{H}}(x \ast y), 0.5 \} \geq \inf \{ \hat{\mathcal{H}}((x \ast y) \ast z), \hat{\mathcal{H}}(z) \} = \inf \{ \hat{\mathcal{H}}(0), \hat{\mathcal{H}}(z) \} \geq \hat{\mathcal{H}}(z). \]

Since \( \mathcal{H} \) is an \( m \)-polar \((\mathcal{E}, \mathcal{E} \vee \mathcal{Q})\)-fuzzy ideal of \( X \), we have

\[ \sup \{ \hat{\mathcal{H}}(x), 0.5 \} \geq \inf \{ \hat{\mathcal{H}}(x \ast y), \hat{\mathcal{H}}(y) \}. \]
Now, 
\[
\begin{align*}
\sup\{\hat{H}(x), 0.5\} & \geq \sup\{\hat{H}(x \ast y), 0.5\} \\
\sup\{\hat{H}(x), 0.5\} & \geq \inf\{\sup\{\hat{H}(x \ast y), 0.5\}, \sup\{\hat{H}(y), 0.5\}\} \\
& \geq \inf\{\hat{H}(x \ast y), \hat{H}(y)\} \\
& = \inf\{\hat{H}(y), \hat{H}(z)\}.
\end{align*}
\]

\[\square\]

**Theorem 3.3.** An m-PF set \(\hat{H}\) of \(X\) is an m-polar \((\overline{\tau}, \overline{\tau} \lor \overline{\eta})\)-fuzzy ideal of \(X\) if and only if \(\hat{H}_\ell \neq \phi\) is an ideal of \(X\) for all \(\ell \in (0.5, 1]^m\).

**Proof.** Assume that \(\hat{H}\) is an m-polar \((\overline{\tau}, \overline{\tau} \lor \overline{\eta})\)-fuzzy ideal of \(X\) and \(\hat{t} \in (0.5, 1]^m\). Suppose \(x \in \hat{H}_\ell\). Then, \(\hat{H}(x) \geq \hat{t}\). Now, 
\[
\sup\{\hat{H}(0), 0.5\} \geq \hat{H}(x) \geq \hat{t}.
\]
Thus, \(\hat{H}(0) \geq \hat{t}\). Hence, \(0 \in \hat{H}_\ell\). Let \(x \ast y, y \in \hat{H}_\ell\). Then, \(\hat{H}(x \ast y) \geq \hat{t}\) and \(\hat{H}(y) \geq \hat{t}\). Now, 
\[
\sup\{\hat{H}(x), 0.5\} \geq \inf\{\hat{H}(x \ast y), \hat{H}(y)\} \geq \hat{t}.
\]
Thus, \(\hat{H}(x) \geq \hat{t}\), that is, \(x \in \hat{H}_\ell\). Therefore, \(\hat{H}_\ell\) is an ideal of \(X\).

Conversely, assume \(\hat{H}_\ell \neq \phi\) is an ideal of \(X\). Let \(x \in X\) be such that \(\sup\{\hat{H}(0), 0.5\} < \hat{H}(x)\). Choose \(\hat{\ell} \in (0.5, 1]^m\) such that 
\[
\sup\{\hat{H}(0), 0.5\} < \hat{\ell} \leq \hat{H}(x).
\]
Then, \(\hat{H}(x) < \hat{\ell}\) and \(x \in \hat{H}_\ell\). Since \(\hat{H}_\ell\) is an ideal of \(X\), we have \(0 \in \hat{H}_\ell\), and so \(\hat{H}(0) \geq \hat{\ell}\), a contradiction. Hence, \(\sup\{\hat{H}(0), 0.5\} \geq \hat{H}(x)\) for all \(x \in X\). Assume \(x, y \in X\) such that \(\sup\{\hat{H}(x), 0.5\} < \inf\{\hat{H}(x \ast y), \hat{H}(y)\}\). Choose \(\hat{t} \in (0.5, 1]^m\) such that 
\[
\sup\{\hat{H}(x), 0.5\} < \hat{t} \leq \inf\{\hat{H}(x \ast y), \hat{H}(y)\}.
\]
Then, \(\hat{H}(x) < \hat{t}\). Since \(x \ast y, y \in \hat{H}_\ell\) and \(\hat{H}_\ell\) is an ideal of \(X\), so \(x \in \hat{H}_\ell\). That is, \(\hat{H}(x) \geq \hat{t}\). This is a contradiction. Thus, \(\sup\{\hat{H}(x), 0.5\} \geq \inf\{\hat{H}(x \ast y), \hat{H}(y)\}\) for all \(x, y \in X\). Hence, \(\hat{H}\) is an m-polar \((\overline{\tau}, \overline{\tau} \lor \overline{\eta})\)-fuzzy ideal of \(X\). \[\square\]

4. **Conclusions**

In this work, first we have introduced the notion of m-polar \((\overline{\tau}, \overline{\tau} \lor \overline{\eta})\) fuzzy ideals and investigated some of its properties. Then, we have discussed the relation between ordinary ideals and m-polar \((\overline{\tau}, \overline{\tau} \lor \overline{\eta})\) fuzzy ideals in BCK/BCI-algebras.

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