GENERALIZED SYMMETRIC BI-DERIVATIONS OF LATTICES

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ABSTRACT. In this article, the notion of a new kind of derivation is introduced for a lattice $L$ called symmetric bi-$(T, F)$-derivations on $L$ as a generalization of derivation of lattices and characterized some of its related properties. Some equivalent conditions provided for a lattice $L$ with greatest element $1$ by the notion of isotone symmetric bi-$(T, F)$-derivation on $L$. By using the concept of isotone derivation, we characterized the modular and distributive lattices by the notion of isotone symmetric bi-$(T, F)$-derivation.

1. INTRODUCTION

The notion of lattice theory first introduced by Birkhof [7]. After the initiation of lattices many researchers studied lattice theory in different point of view such as, Balbes and Dwinger [3] gave the concept on distributive lattices and Hoffmann gave the notion of partially ordered set (Poset). The application of lattice theory plays an important role in different areas such as information theory [4], information retrieval [10], information access controls [39] and cryptanalysis [14]. Recently, the properties of lattices were studied by some authors [13, 16, 26] in analytic and algebraic point of view.

Derivations is a very interesting research area in the theory of algebraic structure in mathematics. Posner [37] provided the concept of derivation on rings. Based on this concept Bell and Kappe [5] studied that rings in which derivations satisfy certain algebraic conditions and Kaya [28] applied the notions of derivations on prime rings. The notion of generalized derivation in ring introduced by Braser [8, 9] and Hvala [17]. This concept of derivation further carried out by many authors [2, 15] in prime rings and lie ideal in prime rings. Jun and Xin [25] applied the notion of derivation in ring and near ring theory to $BCI$-algebra. Later on, Muhiuddin et al. studied the theory of derivations in $BCI$-algebras on different aspects (see for e.g., [31], [32], [33], [34]). Jana et al. [21-30] and Bej and Pal [6] and Senapati et al. [40] has done lot of works on $BCK/BCI$-algebra and $B/BG/G$-algebras which is related to these algebras. Zhan and Liu [45] studied the notion of left-right (respectively, right-left) $f$-derivation of $BCI$-algebras and investigated its properties. The study of derivation in lattice theory is an important topic in application of different mode. Xin et al. [43] introduced the notion of derivation in lattices and discussed its properties. Thereafter, many authors generalized this idea in lattices. For example Yilmaz and Öztürk [44] introduced the notion of $f$-derivation on lattices and its

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some related properties discussed, Çeven [12] studied symmetric bi-derivation on lattice, Kim [27] further investigated symmetric bi-$f$-derivations on lattices, Alshehri [11] studied generalized derivation on lattices and Chaudhry and Ullah [11] introduced the notion of $(\alpha, \beta)$-generalized derivations on lattices and some of its related properties investigated. After symmetric bi-derivation studied by Maksa [29, 30], many researchers introduced this concept to study symmetric bi-derivation on rings and near-rings [35, 36, 38, 41, 42]. Recently, Çeven [12] studied symmetric bi-derivation on lattices and investigated some properties on it. Motivated by the above works and best of our knowledge there is no work available on symmetric bi-$(T, F)$-derivations on lattices. For this reason we developed theoretical study of symmetric bi-$(T, F)$-derivation on lattices.

In this paper, the notion of symmetric bi-$(T, F)$-derivation which is a generalization of derivation in lattices is introduced and studied some properties of it. We gave some equivalent condition for which a derivation to be an isotone symmetric bi-$(T, F)$-derivation for a lattices with greatest element. We characterized modular lattices and distributive lattices by the concept of isotone symmetric bi-$(T, F)$-derivation.

2. Preliminaries

Definition 2.1. [7] Let $L$ be a non-empty set endowed with operations $\wedge$ and $\vee$. Then the set $(L, \wedge, \vee)$ is called lattices if for all $x, y, z \in L$ satisfies the following conditions:

$(L1)$ $x \wedge x = x, x \vee x = x$
$(L2)$ $x \wedge y = y \wedge x, x \vee y = y \vee x$
$(L3)$ $(x \wedge y) \wedge z = x \wedge (y \wedge z), (x \vee y) \vee z = x \vee (y \vee z)$
$(L4)$ $(x \wedge y) \vee x = x, (x \vee y) \wedge x = x$.

Definition 2.2. [7] A Lattice $(L, \wedge, \vee)$ is called distributive lattice if for all $x, y, z \in L$ satisfies the following conditions:

$(L5)$ $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
$(L6)$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

It is notified that in a Lattice the conditions $(L5)$ and $(L6)$ are equivalent.

Definition 2.3. [7] Let $(L, \wedge, \vee)$ be a lattice. A binary relations $(\leq)$ on $L$ defined by $x \leq y$ is holds if and only if $x \wedge y = x$ and $x \vee y = y$.

Definition 2.4. [3] A lattice $(L, \wedge, \vee)$ is called a modular lattice if for all $x, y, z \in L$ satisfies the following conditions:

$(L7)$ If $x \leq y$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Definition 2.5. [43] Let $(L, \wedge, \vee)$ be a lattice. Then $(L, \leq)$ is a poset, i.e. it is a partially ordered set and for any $x, y \in L, x \wedge y$ is the g.l.b of $\{x, y\}$, and $x \vee y$ is the l.u.b of $\{x, y\}$.

Definition 2.6. [7] Let $F : L \rightarrow M$ be a function from the lattice $L$ to the lattice $M$ and is called lattice homomorphism if it satisfies the conditions:

$(L_{10})$ $F(x \wedge y) = F(x) \wedge F(y), F(x \vee y) = F(x) \vee F(y)$ for all $x, y \in L$.

Definition 2.7. [43] Let $L$ be a lattice and $d$ be a self-map on $L$. Then for all $x, y \in L$, $d$ is called derivation on $L$ if satisfying the following identity:

$d(x \wedge y) = (d(x \wedge y) \vee (x \wedge d(y))$

Proposition 2.1. [43] Let $L$ be a lattice and $d$ be a derivation on $L$. Then following conditions are hold:

$(1)$ $d(x) \leq x$
(2) \(d(x) \land d(y) \leq d(x \land y) \leq d(x) \lor d(y)\)

(3) If \(L\) has a least element 0 and a greatest element 1, then \(d(0) = 0\) and \(d(1) = 1\).

**Definition 2.8.** Let \(L\) be a lattice and \(d\) be a derivation on \(L\)

(5) If \(x \leq y\) implies \(d(x) \leq d(y)\), then \(d\) is called an isotone derivation

(6) If \(d\) is one-to-one, then \(d\) is called a monomorphic derivation

(7) If \(d\) is onto, then \(d\) is called an epimorphic derivation.

**Definition 2.9.** Let \((L, \land, \lor)\) be a lattice. A function \(D : L \times L \rightarrow L\) is called symmetric if it satisfies the condition \(D(x, y) = D(y, x)\) for all \(x, y \in L\).

**Definition 2.10.** Let \(L\) be a lattice. A function \(d : L \times L \rightarrow L\) defined by \(d(x) = D(x, x)\) is called trace of \(D\), where \(D\) is a symmetric function.

**Definition 2.11.** Let \(L\) be a lattice and Let \(D : L \times L \rightarrow L\) be a symmetric function on \(L\). Then \(D\) is called symmetric bi-derivation on \(L\) if it satisfies the following identity:

\[D(x \land y, z) = (D(x, z) \land y) \lor (x \land D(y, z))\]

for all \(x, y, z \in L\). Also, a symmetric bi-derivation \(D\) satisfies the following relation

\[D(x, y \land z) = (D(x, y) \land z) \lor (y \land D(x, z))\]

for all \(x, y, z \in L\).

3. Symmetric bi-\((T, F)\)-derivations on lattices

In this section, symmetric bi-\((T, F)\)-derivation on a lattices is introduced.

**Definition 3.1.** Let \(L\) be a lattice. Then for any \(T \in L\), we define a self-map \(D_T : L \times L \rightarrow L\) by \(D_T(x, y) = (x \land y) \land T\) for all \(x, y \in L\).

**Definition 3.2.** Let \(L\) be a lattice. Then for any \(T \in L\), a self-map \(D_T : L \times L \rightarrow L\) is defined as for any \(T \in L\), \(D_T(x, y) = (x \land y) \land T\) for all \(x \in L\). Then then function \(D_T : L \times L \rightarrow L\) is called symmetric bi-\((T, F)\)-derivation of \(L\) if there exist a function \(F : L \rightarrow L\) satisfies the condition:

\[D_T(x \land y, z) = (D_T(x, z) \land F(y)) \lor (F(x) \land D_T(y, z))\]

for all \(x, y, z \in L\). Also, a symmetric bi-\((T, F)\)-derivation \(D_T\) satisfies the following relation

\[D_T(x, y \land z) = (D_T(x, y) \land F(z)) \lor (F(y) \land D_T(x, z))\]

for all \(x, y, z \in L\).

It is notified in the Definition 3.2 that if \(F\) is an identity function then \(D_T\) is a symmetric bi-\(T\)-derivation on \(L\). Therefore, according to Definition 3.2, \(D_T\) is a symmetric bi-\((T, F)\)-derivation on \(L\) if \(F\) must satisfied the identity of the Definition 3.2.

**Example 3.1.** Let \(L = \{0, a, b, 1\}\) be a lattice shown by the Hasse diagram of Figure 1.

For any \(T \in L\), define a self-map \(D_T : L \times L \rightarrow L\) of a lattice \(L\) given in figure 2.

Define the mapping \(D_T\) as follows:

for \(T = 0\), \(D_T(x, y) = 0\) for all \((x, y) \in L \times L\)

for \(T = a\), \(D_T(x, y) = 0\) for all \((x, y) \in \{(0, 0), (0, a), (a, 0), (b, 0), (0, b), (1, 0), (0, 1)\}\)
\[ \mathcal{D}_T(x, y) = a \text{ for all } (x, y) \in \{(a, a), (a, b), (b, a), (a, 1), (b, 1), (1, b), (1, 1)\} \]

for \( T = b \), \( \mathcal{D}_T(x, y) = 0 \) for all \( (x, y) \in \{(0, 0), (a, 0), (0, a), (0, b), (b, 0), (1, 0), (0, 1)\} \),
\[ \mathcal{D}_T(x, y) = a \text{ for all } (x, y) \in \{(a, a), (a, b), (b, a), (a, 1), (1, a)\} \text{ and } \mathcal{D}_T(x, y) = b \text{ for all } (x, y) \in \{(b, b), (b, 1), (1, b)\} \]

For \( T = 1 \), \( \mathcal{D}_T(x, y) = 0 \) for all \( (x, y) \in \{(0, 0), (0, a), (a, 0), (0, b), (b, 0), (1, 0), (0, 1)\} \),
\[ \mathcal{D}_T(x, y) = a \text{ for all } (x, y) \in \{(a, a), (a, b), (b, a), (a, 1), (1, a)\}, \text{ and } \mathcal{D}_T(x, y) = b \text{ for all } (x, y) \in \{(b, b), (b, 1), (1, b)\} \text{ and } \mathcal{D}_T(x, y) = 1 \text{ for } (x, y) = (1, 1). \]

If we define the function \( F \) by \( F(0) = 0, F(a) = a, F(b) = 1 \) and \( F(1) = b \), then it is verified that for each \( T \in L \), \( \mathcal{D}_T \) is a symmetric bi-(\( T, F \))-derivation on \( L \).

Where as, if we defined the function \( F \) by \( F(0) = 0, F(a) = b, F(b) = a \) and \( F(1) = 1 \), then it is justified that for each \( T \in L \), \( \mathcal{D}_T \) is not a symmetric bi-(\( T, F \))-derivation of \( L \), because

for \( T = b \), we have \( \mathcal{D}_T(a \land b, 1) = \mathcal{D}_T(a, 1) = (a \land 1) \land b = a \land b = a \), but
\( (\mathcal{D}_T(a, 1) \land F(b)) \lor (F(a) \land \mathcal{D}_T(b, 1)) \)
\( = (a \land 1) \land (b \land b) \land (F(a) \land F(b)) \)
\( = a \land (b \land b) = a \lor b = b. \) Therefore,
\[ \mathcal{D}_T(a \land b, 1) = a \neq b = (\mathcal{D}_T(a, 1) \land F(b)) \lor (F(a) \land \mathcal{D}_T(b, 1)). \]

**Example 3.2.** Let \( L \) be a lattice and \( T \in L \). Defining a function \( \mathcal{D}_T : L \to L \) by
\( \mathcal{D}_T(x, y) = (F(x) \land F(y)) \land T \) for all \( x, y \in L \) where \( F : L \to L \) satisfying \( F(x \land y) = F(x) \land F(y) \) for all \( x, y \in L \). Then \( \mathcal{D}_T \) is a symmetric bi-(\( T, F \))-derivation of \( L \). In addition, if \( F \) is an increasing function then \( \mathcal{D}_T \) is an isotone symmetric bi-(\( T, F \))-derivation on \( L \).

**Theorem 3.3.** Let \( L \) be a lattice and \( \mathcal{D}_T \) be a trace of symmetric bi-(\( T, F \))-derivation \( \mathcal{D}_T \). Then following conditions are hold for all \( x, y \in L \).

1. \( \mathcal{D}_T(x, y) \leq F(x) \text{ and } \mathcal{D}_T(x, y) \leq F(y) \)
2. \( \mathcal{D}_T(x, y) \land \mathcal{D}_T(w, y) \leq \mathcal{D}_T(x \land w, y) \leq \mathcal{D}_T(x, y) \lor \mathcal{D}_T(w, y) \)
3. \( \mathcal{D}_T(x \land w, y) \leq F(x) \lor F(w) \)
4. \( \mathcal{D}_T(x, y) \leq F(x) \land F(y) \)
5. \( \mathcal{D}_T(x) \leq F(x) \)
6. \( d_T^2(x) = \mathcal{D}_T(x) \).
Proof. (1) Since $D_T(x, y) = D_T(x \land y) = (D_T(x, y) \land F(x)) \lor (F(x) \land D_T(x, y)) = F(x) \land D_T(x, y)$ from which we get $D_T(x, y) \leq F(x)$. Similarly, $D_T(x, y) \leq F(y)$ for all $x, y \in L$.

(2) Since $D_T(x, y) \leq F(x)$ and $D_T(w, y) \leq F(w)$. Then, we have $D_T(x, y) \land D_T(w, y) \leq F(x) \land D_T(x, y)$, and from (1) $D_T(x, y) \land D_T(x, y) \leq F(w) \land D_T(x, y)$ for all $x, y, w \in L$. Hence, $D_T(x, y) \land D_T(w, y) \leq (F(x) \land D_T(x, y)) \lor (F(w) \land D_T(x, y)) = D_T(x \land w, y)$. Also, since $F(x) \land D_T(w, y) \leq D_T(w, y)$ and $F(w) \land D_T(x, y) \leq D_T(x, y)$, and hence obtained $(F(x) \land D_T(x, y)) \lor (F(w) \land D_T(x, y) \leq D_T(x, y) \lor D_T(w, y)$. Thus, $D_T(x \land w, y) \leq D_T(x, y) \lor D_T(w, y)$.

(3) Since $D_T(x, y) \land F(w) \leq F(w)$ and $F(x) \land D_T(x, y) \leq F(x) \lor D_T(x, y)$. Hence, $D_T(x, w) \leq F(x) \lor F(w)$.

(4) From (1) it is clear that $D_T(x, y) \leq F(x) \lor F(y)$ for all $x, y \in L$.

(5) Since $D_T(x) = D_T(x \land x) = (D_T(x, x) \land F(x)) \lor (F(x) \land D_T(x, x)) = F(x) \land D_T(x, y)$, we have obtained $D_T(x) \leq F(x)$ for all $x \in L$.

(6) From (5) it is seen that $d^2_T(x) = D_T(D_T(x)) \leq D_T(x) \leq F(x)$ and also from $(A)$ gives $D_T(x, D_T(x)) \leq D_T(x)$. Then, we have

\[
\begin{align*}
  d^2_T(x) &= D_T(D_T(x)) = D_T(F(x) \land D_T(x)) \\
  &= D_T(F(x), D_T(x)) \lor (F(x) \land d^2_T(x)) \lor (D_T(x) \land F(x)) \\
  &= D_T(F(x), D_T(x)) \lor d^2_T(x) \lor D_T(x) \\
  &= D_T(F(x), D_T(x)) \lor D_T(x).
\end{align*}
\]

Corollary 3.1. Let $L$ be a lattice and $D_T$ be a symmetric bi-(T, F)-derivation on $L$ with least element 0 and greatest element 1, then $F(0) = 0$ and $F(1) = 1$ implies $D_T(0, x) = 0$ and $D_T(1, x) \leq F(x)$ for all $x \in L$.

Proof: The proof of the corollary is trivial by Theorem 3.3 (1).

Theorem 3.4. Let $L$ be a lattice and $D_T$ be symmetric bi-(T, F)-derivation of $L$ and $D_T$ be the trace of symmetric bi-(T, F)-derivation $D_T$. Then,

\[
D_T(x \land y) = D_T(x, y) \lor (F(x) \land D_T(y)) \lor (F(y) \land D_T(x))
\]

for all $x, y \in L$.

Proof. By using the Theorem 3.3 (1) and (5), we have

\[
\begin{align*}
  D_T(x \land y) &= D_T(x \land y, y \land y) \\
  &= (D_T(x \land y, y) \land F(y)) \lor (D_T(x \land y, y) \land F(x)) \\
  &= D_T(x \land y, y) \lor D_T(x \land y, y) \\
  &= ((D_T(x, y) \land F(y)) \lor (F(x) \land D_T(x, y))) \lor (((D_T(x, y) \land F(y)) \lor (F(x) \land D_T(y))) \\
  &= ((D_T(x, y) \land F(y)) \lor (F(x) \land D_T(x, y))) \lor (D_T(x, y) \lor (F(x) \land D_T(y))) \\
  &= D_T(x, y) \lor (F(x) \land D_T(y)) \lor (F(y) \land D_T(x)).
\end{align*}
\]

Corollary 3.2. Let $L$ be a lattice and $D_T$ be symmetric bi-(T, F)-derivation of $L$ and $D_T$ be the trace of symmetric bi-(T, F)-derivation $D_T$. Then the following inequalities hold: for all $x, y \in L$

1. $D_T(x, y) \leq D_T(x, y)$
Proposition 3.5. Let $L$ be a lattice with least element 0 and greatest element 1, and $D_T$ be a symmetric bi-$(T, F)$-derivation of $L$ and $D_T$ be the trace of symmetric bi-$(T, F)$-derivation $D_T$, then following results hold:

1. If $F(x) \geq D_T(1, y)$, then $D_T(x, y) \geq D_T(1, y)$
2. If $F(x) \leq D_T(1, y)$, then $D_T(x) = F(x)$

Proof. (1) Let $F(1) = 1$, then

$$D_T(x, y) = D_T(x \wedge 1, y)$$

$$= (D_T(x, y) \wedge F(1)) \vee (F(x) \wedge D_T(1, y))$$

$$= D_T(x, y) \vee D_T(1, y).$$

Hence, $D_T(x, y) \geq D_T(1, y)$ for all $x, y \in L$.

(2)

$$D_T(x, y) = D_T(x \wedge 1, y)$$

$$= (D_T(x, y) \wedge F(1)) \vee (F(x) \wedge D_T(1, y))$$

$$= D_T(x, y) \vee F(x).$$

Then, $F(x) \leq D_T(x, y)$. Hence by Theorem 3.3(1), $D_T(x, y) = F(x)$ for all $x, y \in L$. 

Theorem 3.6. Let $L$ be a lattice with greatest element 1 and let $D_T$ be a trace of a symmetric bi-$(T, F)$-derivation $D_T$. Then following conditions are equivalent:

1. $D_T$ is an isotone mapping
2. $D_T(x) = F(x) \wedge D_T(1)$
3. $D_T(x \wedge y) = D_T(x) \wedge D_T(y)$
4. $D_T(x) \vee D_T(y) \leq D_T(x \vee y)$

Proof. (1) $\Rightarrow$ (2). Since $D_T$ is isotone and $x \leq 1$, then $F(x) \leq D_T(1)$. Also, $D_T(x) \leq F(x) \wedge D_T(1)$ by Theorem 3.3(E). By Corollary 3.2(B), we have $F(x) \wedge D_T(1) \leq D_T(x)$ for all $x \in L$. Hence, $D_T(x) = F(x) \wedge D_T(1)$ for all $x \in L$.

(2) $\Rightarrow$ (3). Let $F(x) \wedge D_T(1) = D_T(x)$ for all $x \in L$. Then, $D_T(x \wedge y) = F(x \wedge y) \wedge D_T(1) = (F(x) \wedge D_T(1)) \wedge (F(y) \wedge D_T(1)) = D_T(x) \wedge D_T(y)$ for all $x, y \in L$.

(3) $\Rightarrow$ (1). Let $D_T(x \wedge y) = D_T(x) \wedge D_T(y)$ for all $x, y \in L$ and $x \leq y$. Then, $D_T(x) = D_T(x \wedge y) = D_T(x) \wedge D_T(y)$, and hence $D_T(x) \leq D_T(y)$.

(1) $\Rightarrow$ (4). Let $D_T$ be isotone. Since $x \leq x \vee y$ and $y \leq x \vee y$, then $D_T(x) \leq D_T(x \vee y)$ and $D_T(y) \leq D_T(x \vee y)$. Thus, $D_T(x) \vee D_T(y) \leq D_T(x \vee y)$ for all $x, y \in L$.

(4) $\Rightarrow$ (1). Let $x \leq y$, then $D_T(x) \leq D_T(x \vee y) = D_T(y)$. Hence, $D_T$ is isotone. 

Definition 3.3. Let $L$ be a lattice and $D_T$ be a symmetric bi-$(T, F)$-derivation of $L$.

1. If $x \leq w$ implies $D_T(x, y) \leq D_T(w, y)$, then $D_T$ is called an isotone symmetric bi-$(T, F)$-derivation.
2. If $D_T$ is one-to-one, then $D_T$ is called a monomorphic symmetric bi-$(T, F)$-derivation.
3. If $D_T$ is onto, then $D_T$ is called an epic symmetric bi-$(T, F)$-derivation.
Lemma 3.7. Let $D_T$ be a symmetric bi-$(T, F)$-derivation on lattice $L$. Then followings hold:

1. $D_T(x \land w, y) = D_T(x, y) \land D_T(w, y)$ for all $x, y, w \in L$
2. $D_T(x \lor w, y) \geq D_T(x, y) \lor D_T(w, y)$ for all $x, y, w \in L$.

Proof. (1) Since $x \land w \leq x$ and $x \land w \leq w$, then we have $D_T(x \land w, y) \leq D_T(x, y)$ and $D_T(x \land w, y) \leq D_T(w, y)$. Thus $D_T(x \land w, y) \leq D_T(x, y) \land D_T(w, y)$. Hence, by Theorem 3.3(2), we get $D_T(x \land w, y) = D_T(x, y) \land D_T(w, y)$ for all $x, y, w \in L$.

(2) Since $x \leq x \lor y$ and $y \leq x \lor y$, so we have $D_T(x, y) \leq D_T(x \lor w, y)$ and $D_T(w, y) \leq D_T(x \lor w, y)$. Therefore, we obtained $D_T(x \lor w, y) \geq D_T(x, y) \lor D_T(w, y)$ for all $x, y, w \in L$. □

Proposition 3.8. Let $L$ be a lattice and $D_T$ be a symmetric bi-$(T, F)$-derivation on $L$. Then followings hold:

1. $D_T(x, y) = D_T(x, y) \lor (D_T(x \lor s, y) \land x)$, when $D_T$ is an symmetric bi-$(T, F)$-derivation on $L$
2. $D_T(x, y) = D_T(x, y) \lor (D_T(x \lor s, y) \land F(x))$, when $F$ is a join-homomorphism on $L$
3. Then $D_T(x, y) = D_T(x, y) \lor (F(x) \land D_T(x \lor s, y))$, when $F$ is an increasing function on $L$.

Proof. (1) Let $D_T$ be an isotone symmetric bi-$(T, F)$-derivation. Then,

$$D_T(x, y) = D_T((x \lor s) \land x, y)$$
$$= (D_T(x \lor s, y) \land F(x)) \lor (F(x \land s) \land D_T(x, y))$$
$$= (D_T(x \lor s, y) \land F(x)) \lor D_T(x, y).$$

As, $D_T(x, y) \leq D_T(x \lor s, y) \leq F(x \lor s)$.

(2) Since $D_T(x, y) \leq F(x) \leq F(x) \lor F(s)$ and $F(x \lor s) = F(x) \lor F(s)$, so obtained

$$D_T(x, y) = D_T((x \lor s) \land x, y)$$
$$= (D_T(x \lor s, y) \land F(x)) \lor (F(x \land s) \land D_T(x, y))$$
$$= (D_T(x \lor s, y) \land F(x)) \lor D_T(x, y).$$

(3) Since $F$ is an increasing function and $x \leq x \lor y$, so $F(x) \leq F(x \lor y)$. Therefore,

$$D_T(x, y) = D_T((x \lor s) \land x, y)$$
$$= (D_T(x \lor s, y) \land F(x)) \lor (F(x \land s) \land D_T(x, y))$$
$$= (D_T(x \lor s, y) \land F(x)) \lor D_T(x, y).$$

□

Theorem 3.9. Let $L$ be a lattice with greatest element 1 and $D_T$ be a symmetric bi-$(T, F)$-derivation on $L$ and $F(x \land y) = F(x) \land F(y)$. Then followings equivalent:

1. $D_T$ is isotone symmetric bi-$(T, F)$-derivation
2. $D_T(x, y) \lor D_T(s, y) \leq D_T(x \lor s, y)$ for all $x, y \in L$
3. $D_T(x, y) = F(x) \land D_T(1, y)$ for all $x, y \in L$
4. $D_T(x \land s, y) = D_T(x, y) \land D_T(s, y)$ for all $x, y, s \in L$.

Proof. (1) $\Rightarrow$ (2). We assume that $D_T$ is an isotone symmetric bi-$(T, F)$-derivation on $L$. Since $x \leq x \lor s$ and $s \leq x \lor s$, and so $D_T(x, y) \leq D_T(x \lor s, y)$ and $D_T(s, y) \leq D_T(x \lor s, y)$. Hence, $D_T(x, y) \lor D_T(s, y) \leq D_T(x \lor s, y)$ for all $x, y, s \in L$. □
Since, \( \text{symmetric bi-} \Rightarrow \)

Let \((1)\)

Let \(3.8\) and since

Theorem 3.11. Let \(L\) be a modular lattice and \(D_T\) be a symmetric bi-\((T, F)\)-derivation on \(L\). Then, followings hold.

(1) If \(D_T\) is an isotone symmetric bi-\((T, F)\)-derivation on \(L\) and only if \(D_T(x \land s, y) = D_T(x, y) \land D_T(s, y)\)

(2) If \(D_T\) is an isotone symmetric bi-\((T, F)\)-derivation on \(L\) and \(F(x \lor s) = F(x) \lor F(s)\),

\(D_T(x, y) = F(x)\), then \(D_T(x \land s, y) = D_T(x, y) \land D_T(s, y)\).

Proof. (1) Let \(D_T\) be a symmetric bi-\((T, F)\)-derivation on \(L\). Since \(x \land s \leq x\) and \(x \land s \leq s\), then \(D_T(x \land s, y) \leq D_T(x, y) \land D_T(s, y)\). Therefore,

\[
D_T(x, y) \land D_T(s, y) = (D_T(x, y) \land D_T(s, y)) \land (F(x) \land F(s))
\]

\[
= (D_T(x, y) \land F(s)) \land F(x) \land D_T(s, y))
\]

\[
\leq (D_T(x, y) \land F(s)) \lor (D_T(s, y) \land F(x))
\]

\[
= D_T(x \land s, y).
\]

Conversely, let \(D_T(x \land s, y) = D_T(x, y) \land D_T(s, y)\) and \(x \leq s\). Thus, \(D_T(x, y) = D_T(x \land s, y) = D_T(x, y) \land D_T(s, y)\), and hence \(D_T(x, y) \leq D_T(s, y)\) for all \(x, s \in L\).

(2) Let \(D_T\) be a symmetric bi-\(T\)-derivation on \(L\) and \(D_T(x, y) = x\). Then, by Proposition \(3.8\) and since \(L\) is a modular lattice, thus, \(D_T(s, y) = (D_T(s, y) \lor D_T(x \land s, y)) \lor F(s) = F(s) \land D_T(x \lor s, y)\).

Thus,

\[
D_T(x, y) \lor D_T(s, y) = D_T(x, y) \lor (F(s) \land D_T(x \land s, y))
\]

\[
= (D_T(x, y) \lor F(s)) \land D_T(x \land s, y)
\]

\[
= (F(x) \lor F(s)) \land D_T(x \land s, y)
\]

\[
= F(x \lor s) \land D_T(x \lor s, y)
\]

\[
= D_T(x \lor s, y).
\]

\(\square\)

Theorem 3.11. Let \(L\) be a distributive lattice and \(D_T\) be a symmetric bi-\((T, F)\)-derivation on \(L\), and \(F(x \lor s) = F(x) \lor F(s)\). Then, following conditions are hold.

(A) If \(D_T\) is an isotone symmetric bi-\((T, F)\)-derivation on \(L\), then \(D_T(x \land s, y) = D_T(x, y) \land D_T(s, y)\)

(B) If \(D_T\) is an isotone symmetric bi-\((T, F)\)-derivation on \(L\) and only if \(D_T(x \lor s, y) = D_T(x, y) \lor D_T(s, y)\).
Proof. Since, $\mathcal{D}_T$ is an isotone symmetric bi-$T$-derivation and $\mathcal{D}_T(x \land s, y) = \mathcal{D}_T(x, y) \land \mathcal{D}_T(s, y)$. By Theorem 3.3 (A), we have
\[
\mathcal{D}_T(x, y) \land \mathcal{D}_T(s, y) = (((\mathcal{D}_T(x, y) \land F(x)) \land (F(s) \land \mathcal{D}_T(s, y)))
\]
\[
= (\mathcal{D}_T(x, y) \land F(s)) \land (F(x) \land \mathcal{D}_T(s, y))
\]
\[
\leq (\mathcal{D}_T(x, y) \land F(s)) \lor (F(x) \land \mathcal{D}_T(s, y))
\]
\[
= \mathcal{D}_T(x \land s, y).
\]
Therefore, $\mathcal{D}_T(x \land s, y) = \mathcal{D}_T(x, y) \land \mathcal{D}_T(s, y)$ for all $x, y, s \in L$. (B) Let $\mathcal{D}_T$ be an isotone symmetric bi-$T$-$F$-derivation. Then, using Theorem 3.3 (A) and Proposition 3.8, we have
\[
\mathcal{D}_T(s, y) = (\mathcal{D}_T(s, y) \land (F(s) \land \mathcal{D}_T(x \lor s, y)))
\]
\[
= (\mathcal{D}_T(s, y) \land F(s)) \land (\mathcal{D}_T(s, y) \lor \mathcal{D}_T(x \lor s, y))
\]
\[
= F(s) \land \mathcal{D}_T(x \lor s, y).
\]
In similar way, $\mathcal{D}_T(x, y) = F(x) \land \mathcal{D}_T(x \lor s, y)$. Thus,
\[
\mathcal{D}_T(x, y) \lor \mathcal{D}_T(s, y) = (F(x) \land \mathcal{D}_T(x \lor s, y)) \lor (F(s) \land \mathcal{D}_T(x \lor s, y))
\]
\[
= (F(x) \land F(s)) \land \mathcal{D}_T(x \lor s, y)
\]
\[
= F(x \lor s) \land \mathcal{D}_T(x \lor s, y)
\]
\[
= \mathcal{D}_T(x \lor s, y).
\]
Conversely, let $\mathcal{D}_T(x \lor s, y) = \mathcal{D}_T(x, y) \lor \mathcal{D}_T(s, y)$ and $x \leq s$, then obtained $\mathcal{D}_T(s, y) = \mathcal{D}_T(x \lor s, y) = \mathcal{D}_T(x, y) \lor \mathcal{D}_T(s, y)$, which imply $\mathcal{D}_T(x, y) \leq \mathcal{D}_T(s, y)$ for all $x, y, s \in L$.

\[
\square
\]

4. CONCLUSIONS AND FUTURE WORK

In this paper, we discussed the notion of symmetric bi-(T, F)-derivation on lattice and investigated some useful properties of it. In our opinion, these results can be similarly extended to the other algebraic structure such as $BCI$-algebras, $B$-algebras, $BG$-algebras, $BF$-algebras, $MV$-algebras, $d$-algebras, $Q$-algebras, Incline algebras and so forth. The study of symmetric bi-(T, F)-derivation on different algebraic structures may have a lot of applications in different branches of theoretical physics, engineering, information theory, information retrieval, information control access, cryptanalysis and computer science, etc.

We hope that this work will give a deep impact on the upcoming research in this field and other algebraic study to open up a new horizons of interest and innovations. It is our hope that this work would serve as a foundation for further study in the theory of derivations of lattice.

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